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Mathematical Reviews

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HISTORY

Hofmann, Jos. E. Zur Entdeckungsgeschichte der höheren Analysis im 17. Jahrhundert. Math.-Phys. Semesterber. 1, 220–255 (1950).

Polubarinova-Kočina, P. Ya. On the history of the problem of the rotation of a rigid body. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1949**, 626–632 (1949). (Russian)

Saxer, Walter. Über die Entwicklung des zentralen Grenzwertsatzes der Wahrscheinlichkeitsrechnung. Elemente der Math. **5**, 50–55 (1950).

Rényi, Alfréd. 30 years of mathematics in the Soviet Union. II. New lines of research in probability theory. I. Mat. Lapok **1**, 91–137 (1950). (Hungarian. Russian and English summaries)

Survey of achievements in probability theory by mathematicians working in the USSR during the last 30 years. Two more parts are to follow.

Van Dantzig, D. Blaise Pascal and the significance of the mathematical way of thought for the study of human society. Euclides, Groningen **25**, 203–232 (1950). (Dutch)

Lecture given in October 1948 at the University of Amsterdam. Cf. also a book with the same title, Noordhoff, Groningen, 1949; these Rev. **10**, 500.

Sadykov, H. U. Biruni and his astronomical work. Akad. Nauk SSSR. Astr. Zhurnal **27**, 73–80 (1950). (Russian)

This paper sketches the life and scientific activity of Abū Raiḥān Muḥammad ibn Aḥmad al-Bīrūnī (973–1048), whose birthplace is in the Uzbek SSR. The author first outlines the development of medieval Moslem astronomy, remarking that many individuals known to the European literature as "Arabs" were Central Asiatics whose use of Arabic was analogous to their Western contemporaries' employment of Latin. He then mentions Bīrūnī's main works, emphasizing his discussion of the heliocentric hypothesis and his courageous insistence on the separation of religion and science in a religion-dominated age. As examples of Bīrūnī's theoretical work the author gives his method of computing the longitude of the solar apogee (based on Ptolemy's method, but making use of the sine function unknown in Ptolemaic times) and his method of computing the circumference of the earth.
E. S. Kennedy (Beirut).

✱**Ore, Oystein. Niels Henrik Abel.** Elemente der Math. Beiheft no. 8. Verlag Birkhäuser, Basel, 1950. 23 pp. 3.50 Swiss francs.

García Tranque, T. Biography: Appollonius of Pergama. Gaceta Mat. (1) **1**, 3–10 (1949). (Spanish)

✱**Fleckenstein, J. O. Johann und Jakob Bernoulli.** Elemente der Math. Beiheft no. 6. Verlag Birkhäuser, Basel, 1949. 24 pp. 3.50 Swiss francs.

The seventieth birthday of Sergei Natanovič Bernštein. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 193–198 (1 plate) (1950). (Russian)

A short discussion of S. N. Bernštein's mathematical work during the last ten years and a list, by years, of his published papers from 1940 through 1949. [His publications through 1939 were covered in an earlier article by Gontcharoff and Kolmogoroff, Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 249–260 (1940); these Rev. **2**, 114.]

✱**Schilpp, Paul Arthur, editor. Albert Einstein: Philosopher-Scientist.** The Library of Living Philosophers, Inc., Evanston, Ill., 1949. xvi+781 pp. (2 plates). $8.50.

in Phys

The plan of this book is as follows. First comes an essay on his own work, "something like my own obituary" as Einstein describes it. Next follow essays by twenty-five different authors on various aspects of his achievements, and finally a "reply to criticisms" by Einstein. Lastly there is a bibliography of Einstein's writings to October 1949 compiled by Margaret C. Shields and a voluminous index.

The twenty-five essayists must have found their task a difficult one; were they merely to write laudations or embark on surveys in which no criticisms were barred? Some half-dozen of them have avoided the dilemma by writing, essentially, technical expositions of points in relativity itself or in alternative theories. Such are, for example, the contributions of Robertson or of Milne. Most of the writers are, however, concerned with more contentious philosophical questions or with explaining their own and Einstein's views on the nature of mathematical physics. References to "physical reality" are numerous but the meaning of this term was not clear to the reviewer who suspected that there were as many meanings as there were authors. Yet this seems to be the main question in elucidating Einstein's views about the nature and conclusions of mathematical physics. To Einstein the "laws of nature" appear to be free inventions of the human mind, confirmed by experience, and so, if freely invented, presumably alterable by an act of will. But on the other hand the concepts of physics, such as electrons, light, electromagnetic and gravitational fields, etc., are not likewise free inventions but "realities" whose properties are being discovered. Thus the dual character of light or an electron (particle and wave) appears to be as real a problem for Einstein as for the quantumist group of contributors such as Bohr, Born, or Pauli. If an electron, for instance, is real, it is indeed necessary to know how it can have both a particle and a wave character; if it is a concept invented by the human mind it may well be advantageous to invent it, in some contexts as a particle, and in others as a wave. This dualism in Einstein's own thought is the root difficulty with which the "philosophical" contributors have to contend. Of this group, which includes men like Dingle, Lenzen, Margenau and Ushenko, it is

Bridgman who, on page 347, most clearly indicates the dilemma.

A curious feature of Einstein's own contributions to this book is his apparent imperviousness to the after-effects of that work in quantum theory and in relativity which he did between 1901 and 1916. In the two relativity theories he has shown that one may create new systems of mechanics different from Newton's by starting from premises other than those which Newton employed. Moreover, if the laws of mechanics are inventions of the human mind confirmed by experience, then it is not surprising that the advent of a new field of experience, which we now call atomic physics, might require the invention of still another set of laws. But Einstein will have none of this; he dislikes discontinuity and quantum theory and still strives to establish a unified field theory of gravitation and electromagnetism in which continuity is to be preserved. *G. C. McVittie.*

Fueter, Rudolf. **Leonhard Euler.** Elemente der Math. Beiheft no. 3. Verlag Birkhäuser, Basel, 1948. 24 pp. 3.50 Swiss francs.

Notice biographique: Robert Feys. Synthèse 7, 447–452 (1949).

Kollros, Louis. **Evariste Galois.** Elemente der Math. Beiheft no. 7. Verlag Birkhäuser, Basel, 1949. 24 pp. 3.50 Swiss francs.

Beaujouan, Guy. Documents nouveaux concernant Lagrange. Rev. Hist. Sci. Appl. 3, 110–132 (1950).

Hadamard, J. Célébration du deuxième centenaire de la naissance de P. S. Laplace. Arch. Internat. Hist. Sci. (N.S.) 3, 287–290 (1950).

Hofmann, Jos. E. Nicolaus Mercator (Kauffman), sein Leben und Wirken, vorzugsweise als Mathematiker. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. 1950, no. 3, 45–103 (1950).

Tabuenca Orallo, L. Biography: Nicomacus of Gerasa. Gaceta Mat. (1) 1, 257–262 (1949). (Spanish)

Burckhardt, J. J. **Ludwig Schläfli.** Elemente der Math. Beiheft no. 4. Verlag Birkhäuser, Basel, 1948. 23 pp. 3.50 Swiss francs.

Kollros, Louis. **Jakob Steiner.** Elemente der Math. Beiheft no. 2. Verlag Birkhäuser, Basel, 1947. 24 pp. 3.50 Swiss francs.

Brusotti, Luigi. Obituary: Luigi Berzolari. Boll. Un. Mat. Ital. (3) 5, 1–19 (1 plate) (1950).

Carlson, F. Obituary: Torsten Carleman. Acta Math. 82, I–VI (1 plate) (1950).

Ullrich, Egon. Ein Nachruf auf Friedrich Engel. Mitt. Math. Sem. Univ. Giessen 34, i+14 pp. (1945).

Bohr, Harald. Obituary: Johannes Hjelmslev (1873–1950). Acta Math. 83, vii–ix (1950).

Calapso, Renato. Obituary: Giuseppe Marletta. Atti Accad. Gioenia Catania (6) 6 (1943/49), 24 pp. (1 plate) (1950).

Archibald, Raymond Clare. Obituary: R. G. D. Richardson (1878–1949). Bull. Amer. Math. Soc. 56, 256–265 (1950).

Runge, Iris. Carl Runge und sein wissenschaftliches Werk. Abh. Akad. Wiss. Göttingen. Math.-Phys. Kl. (3) no. 23, 214 pp. (1 plate) (1949).

Järnefelt, G. Obituary: Karl F. Sundman. Acta Math. 83, i–vi (1950).

Cinquini, S. Obituary: Leonida Tonelli. Ann. Scuola Norm. Super. Pisa (2) 15 (1946), 1–37 (1950).

FOUNDATIONS

Skolem, Th. The logical paradoxes and remedies for them. Norsk. Mat. Tidsskr. 32, 2–11 (1950). (Norwegian)

This is an expository article. The principal topics treated are: the Russell paradox and various closely related ones, including that of the library catalog; the avoidance of these through the simplified theory of types; the ancestral relation and its relation to mathematical induction on the one hand, and to the impredicativeness question on the other; recursive arithmetic. The Burali-Forti and Richard paradoxes are barely mentioned at the end. The author stresses the point that mathematics can be founded in various ways, and that in some of these ways the natural numbers are primitive, in others not.

H. B. Curry (Louvain).

Sobociński, Bolesław. L'analyse de l'antinomie russellienne par Leśniewski. III. Methodos 1, 308–316 (1949).

This is the third installment of the author's exposition of Leśniewski's treatment of the Russell paradox [for the other two see the same vol., 94–107, 220–228 (1949); these Rev. 11, 73, 412]. It contains some consequences and alternative forms of previous assumptions. It is difficult to appraise the significance of these details until the final installment appears. *H. B. Curry* (Louvain).

Halldén, Sören. **Några resultat i modal logik.** [Some Results in Modal Logic]. Thesis, Stockholms Högskola, 1950. 34 pp.

This is a report on five previous papers [(1) Norsk. Mat. Tidsskr. 31, 4–9 (1949); (2) J. Symbolic Logic 13, 138–139 (1948); (3) ibid. 14, 230–236 (1950); (4) Theoria 14, 265–269 (1948); (5) Norsk. Mat. Tidsskr. 31, 89–94 (1949); these Rev. 10, 585, 229; 11, 303, 304, 411]. The report contains somewhat more discussion than the original papers, and more exposition of background. It also adds some details to the results; of these the principal items are as follows. A relationship between the systems S3, S4, and S7 in (3) is extended to analogous systems S2.5, S2.6, and S6.5 in which the characteristic postulate of S3 is weakened from a strict to a material implication. In (5) the property of transformations T_1 and T_2, viz., that they map the classical algebra on S5, is extended to T_3; so that we have a complete analogy between the relationship of the Heyting algebra to S4 on

the one hand, and that of the classical algebra to S5 on the other. *H. B. Curry* (Louvain).

Rosser, J. B., and Turquette, A. R. A note on the deductive completeness of m-valued propositional calculi. J. Symbolic Logic **14**, 219–225 (1950).

The authors give a set of axiom schemata for n-valued logic, which are simpler than the ones they gave in an earlier paper [same J. **10**, 61–82 (1945); these Rev. **7**, 185]. The simplification is effected by reducing the number of primitive symbols (by the elimination of conjunction and disjunction), which in turn leads to a considerable reduction in number and complexity of the axiom schemata.

J. C. C. McKinsey (Santa Monica, Calif.).

***Kershner, R. B., and Wilcox, L. R. The Anatomy of Mathematics.** The Ronald Press Company, New York, N. Y., 1950. xi+416 pp. $6.00.

This book gives a precise description of the axiomatic method, illustrated by applications to some fundamental theories. Its point of view can be characterized as follows. Elementary set theory is assumed as intuitively clear; such concepts as the set of all subsets of a given set are used freely. Ordinary logic is applied without discussion. Thus the modern investigations in the foundations of mathematics are ignored, but the treatment of the axiomatic method in its more classical form is clear and instructive. Completely detailed proofs are given throughout. Stress is laid on points of fundamental interest such as inductive definition, the rôle of the axiom of choice, categorical systems of axioms, isomorphism of theories, and equivalence relations. The subject matter comprises groups, positive integers, the theory of cardinal numbers, including Bernstein's equivalence theorem, rational and real numbers, and the theory of fields. A great number of well chosen exercises are given, with suggestions for their solution. *A. Heyting.*

Cassina, Ugo. Le dimostrazioni in matematica. Ann. Mat. Pura Appl. (4) **29**, 131–146 (1949).

This is an exposition of various ideas connected with the technique of demonstration. The standpoint is that of the Peanese school. *H. B. Curry* (Louvain).

da Silva Paulo, J. Axiomatics of Peano. Demonstration of the properties of addition and multiplication by the method of induction. Gaz. Mat., Lisboa **10**, no. 41–42, 22–32 (1949). (Portuguese)

Ulm, Helmut. Konstruktion und deduktive Charakterisierung der Zahlkörper der Analysis. Math.-Phys. Semesterber. **1**, 268–272 (1950).

Borel, Émile. Analyse et géométrie euclidiennes. C. R. Acad. Sci. Paris **230**, 1989–1990 (1950).

Je propose de donner le nom d'analyse euclidienne à l'ensemble des théorèmes d'analyse et de théorie des ensembles et des fonctions que l'on obtient en admettant sans restriction la définition euclidienne de l'égalité des figures géométriques finies, à savoir que deux figures égales sont superposables . . . . Cette définition euclidienne de l'égalité entraîne des conséquences en contradiction avec l'axiome de Zermelo. On pourra donc dire que l'ensemble des théorèmes obtenus par les mathématiciens qui admettent l'axiome de Zermelo constitue une analyse non euclidienne.

Extract from the paper.

Moon, Parry, and Spencer, Domina Eberle. A geometric treatment of "dimensions" in physics. Canadian J. Research. Sect. A. **28**, 268–280 (1950).

A new formulation of dimension theory, in terms of an abstract "idon space," is given. The decision as to which "idon space" is relevant to a given problem is to be determined by each investigator, using criteria described by the authors in another paper [J. Franklin Inst. **248**, 495–521 (1949)]. It is shown that, using an idon space with an extra dimension, one gets a more precise formula for thermionic emission than by the "Pi theorem" with the fundamental units L, M, T, Q.

G. Birkhoff (Cambridge, Mass.).

***Margenau, Henry. The Nature of Physical Reality. A Philosophy of Modern Physics.** McGraw-Hill Book Co., Inc., New York, N. Y., 1950. xiii+479 pp. $6.50.

The object of this book is to provide an adequate basis for, and a consistent interpretation of, both classical and quantum physical theory. Specifically, the attempt is made to resolve the difficulties of physical interpretation by developing a suitable epistemology. The author attributes the common "paradoxes" of microphenomena to the illusions of realism. On the other hand, he regards empiricism as insufficient. His problem is to discover in what sense thing and property are real and compatible with observation. Bare sense perceptions are immediately given. By reification, abstraction, and other rules of correspondence (epistemic correlations), one associates with bare sense perceptions what the author calls constructs, e.g., trees, electrons, mirages, ghosts, the ether, and mass. The philosophical nature of a construct is indicated by the following statements. "The tree in front of my window is a real tree. . . . The tree is the construct. . . . There is not a tree and my construct of it" [pp. 47, 70]. Evidently the rules of correspondence are not uniquely determined for us, for some constructs turn out to be not valid. Criteria are given for validating constructs. The view is taken that "the real world comprises all valid constructs and that part of [sense perceptions] which stands . . . in epistemic correlation with them" [p. 299]. By thus denying realism, it becomes meaningful to make such statements as "particles do not have unique positions in space at specific instants." The difficulty arising from the circumstance that position is commonly thought to be a property of a particle is resolved by interpreting position as a latent observable. A detailed commentary, elaborating this philosophy and showing the significance of both empirical and rational concepts, is given for various classical and quantum theories. A typical feature of this commentary is that the uncertainty principle is developed, not from "a few experiments" with a "pretense of drawing conclusions," but as a direct consequence of the basic methodology of quantum mechanics.

C. C. Torrance (Annapolis, Md.).

ALGEBRA

***Littlewood, D. E. The Skeleton Key of Mathematics. A Simple Account of Complex Algebraic Theories.** Hutchinson's University Library, London; Longmans, Green and Co., Inc., New York, N. Y., 1949. 138 pp. $1.60.

A concise exposition of the ideas and methods of a number of algebraic theories, written in such a way as to be of particular interest to the mathematician who is not a specialist in these fields. Among the topics treated are theory of numbers, algebraic numbers, groups, the Galois theory of equations, coordinate geometry, matrices and determinants, invariants and tensors, group algebras, and group characters.

N. H. McCoy (Northampton, Mass.).

Radhakrishna Rao, C. On a class of arrangements. Proc. Edinburgh Math. Soc. (2) 8, 119–125 (1949).

The author considers vectors $(x_1, \cdots, x_n)$ where each x_i can take any of the values $1, \cdots, s$. There are s^n such vectors. An array of strength d denoted by (N, n, s, d) is a set of N such vectors, all different, such that for any combination $i_1, \cdots, i_d$ and any set of values $a_1, \cdots, a_d$ $(1 \leqq a_i \leqq s)$ there are exactly h vectors $(N = hs^d)$ in (N, n, s, d) for which $x_{i_1} = a_1, \cdots, x_{i_d} = a_d$. The author proves the inequalities

$$N \geqq {}^nC_0 + {}^nC_1(s-1) + \cdots + {}^nC_{\frac{1}{2}d}(s-1)^{\frac{1}{2}d}$$

if d is even,

$$N \geqq {}^nC_0 + {}^nC_1(s-1) + \cdots + {}^nC_{\frac{1}{2}(d-1)}(s-1)^{\frac{1}{2}(d-1)} + {}^{n-1}C_{\frac{1}{2}(d-1)}(s-1)^{\frac{1}{2}(d+1)}$$

if d is odd.

The $(s^2, n, s, 2)$ is equivalent to a set of $n-2$ orthogonal Latin squares. The author gives methods of construction of the arrays $(s^r, n, s, 2)$, $n \leqq (s^r-1)/(s-1)$, when s is a prime power and shows that $(s^r-1)/(s-1)$ is the maximum number possible in this case; and of the arrays $(2^r, n, 2, 3)$, $n \leqq 2^{r-1}$, where 2^{r-1} is the maximum number possible. In general, the solutions of $n-r$ linearly independent equations in n unknowns in the Galois field of order s supply an (s^r, n, s, d) if no equation in the set derived from them has less than $d+1$ nonzero coefficients. For sets derivable in this manner $d \leqq n(s^{n-r} - s^{n-r-1})/(s^{n-r}-1) - 1$.

H. B. Mann (Berkeley, Calif.).

Fisher, R. A. A class of enumerations of importance in genetics. Proc. Roy. Soc. London. Ser. B. 136, 509–520 (1950).

The enumerations given, of isomorphic genotypes, modes of gamete formation, etc., are all of the form $\sum a_k b_k{}^l$, where l is given by the problem, and hence require only a method of determining a_k, b_k. This is done by a double entry table with rows and columns described by partitions of the same number. The partitions of the rows describe the character (likeness or unlikeness) of objects being distributed, while those of the columns describe the cycle structure of a permutation group. The entries of the tables are of the number of distributions of given character which remain invariant under the permutations of the given group, and their sums by columns are the numbers b_k. The a_k are simply the ratios of the numbers of the corresponding permutation groups to the totals of groups. The enumeration of genotypes is said to be of partitions in (on a lattice of) l dimensions, and it is noted that the corresponding b_k are related to the exponential numbers generated by $\exp(e^x-1)$ [but it appears that in the statement of this relationship, the word "factor" should be replaced by "divisor"].

J. Riordan (New York, N. Y.).

Gavrilov, M. A. On a general method of transformation of relay-contact schemes. Avtomatika i Telemehanika 8, 89–107 (1947). (Russian)

A large number of identities for Boolean functions are presented and are interpreted in terms of switching networks. Under special circumstances such identities can be used to transform a switching network into an equivalent network using fewer contacts.

E. N. Gilbert.

Lewis, F. A. On a determinant of Sylvester. Amer. Math. Monthly 57, 324–326 (1950).

Let $\epsilon = e^{2\pi i/n}$, $v_{rc} = \epsilon^{(r-1)(c-1)}$, and $V = |v_{rc}|$, $r, c = 1, 2, \cdots, n$ [see Sylvester, Mathematical Papers, vol. II, p. 621]. The author shows that V is reducible, and, by using a result of Frame [Amer. Math. Monthly 46, 304–305 (1939)], obtains the values of the two component determinants and evaluates certain determinants of the forms $|2\cos(2\pi rc/n)|$ and $|2i\sin(2\pi rc/n)|$ (e.g., $r, c = 1, 2, \cdots, (n-1)/2$ if n is odd).

G. B. Price (Lawrence, Kan.).

Schwartz, A., and de Wet, J. S. The minors of a determinant in terms of Pfaffians. Proc. Cambridge Philos. Soc. 46, 519–520 (1950).

The authors derive a formula by means of which any minor of any determinant (skew-symmetric or not) may be expressed as a polynomial whose terms are products of Pfaffian aggregates of its elements.

L. M. Blumenthal.

Wielandt, Helmut. Unzerlegbare, nicht negative Matrizen. Math. Z. 52, 642–648 (1950).

An indecomposable nonnegative matrix is here one which has nonnegative real elements and which cannot be written in the form

$$\begin{pmatrix} P & R \\ O & Q \end{pmatrix}$$

with P and Q square by making the same permutation on the rows and the columns. For such a matrix A the following results hold. (1) The characteristic equation of A has a simple positive root r of maximum absolute value among the roots. (2) The root r can be characterized in three equivalent ways: (i) $r = \max_{x_\mu \geqq 0} \min_\mu (Ax)_\mu / x_\mu$, (ii) $r = \min_{x_\mu > 0} \max_\mu (Ax)_\mu / x_\mu$, and (iii) if $Ax = \alpha x$ and $x_\mu > 0$ for all μ, $\alpha = r$. (3) If there are exactly k roots of the characteristic equation of A of absolute value r, they are simple and the set of all the roots is invariant under rotation of the complex plane by $2\pi/k$. (4) By a renumbering of the axes, A transforms vectors of the form

$$(0, \cdots, 0, x_{rk+1}, \cdots, x_{(r+1)k}, 0, \cdots, 0)$$

into those of the form $(0, \cdots, 0, y_{(r-1)k+1}, \cdots, y_{rk}, 0, \cdots, 0)$, where the order of A is kl, $r = 1, \cdots, l$ and the subscripts are taken mod kl. (5) If B has complex elements $\beta_{\mu\nu}$ and $|\beta_{\mu\nu}| \leqq a_{\mu\nu}$ for all μ, ν, then $|\beta| \leqq r$ for an arbitrary eigenvalue β of B. (6) If $|\beta| = r$, i.e., $\beta = re^{i\varphi}$, then $B = e^{i\varphi} DAD^{-1}$ with D diagonal with diagonal elements of absolute value one.

W. Givens (Knoxville, Tenn.).

Todd, J. A. On syzygies connecting the covariant line-complexes of two quadric surfaces. Proc. London Math. Soc. (2) 51, 325–347 (1950).

In a previous paper [Proc. Cambridge Philos. Soc. 44, 186–195 (1948); these Rev. 10, 178] the author obtained a number of syzygies connecting the 16 covariant line complexes of the irreducible system of two quaternary quadrics. In this paper he uses different methods to obtain a complete account of all the syzygies of these line complexes. These methods employ the *S*-functional analysis introduced by the reviewer [Philos. Trans. Roy. Soc. London. Ser. A. 239, 305–365 (1944); these Rev. 6, 41]. By intricate arguments of an essentially combinatorial character the author shows that for line complexes of even order all the syzygies are derivable from those obtained in his previous paper, while for complexes of odd order there exist in addition three new syzygies, which are exhibited. *D. E. Littlewood.*

Bulatović, Z. Über die Zurückführung der Auflösung der Gleichung vierten Grades auf die Auflösung einer Gleichung dritten Grades. Bull. Soc. Math. Phys. Serbie 1, no. 3–4, 131–132 (1949). (Serbian. German summary)

Borofsky, Samuel. Factorization of polynomials. Amer. Math. Monthly 57, 317–320 (1950).

Let D be an integral domain and $D[x]$ the integral domain of all polynomials in the indeterminate x with coefficients in D. The author gives a simple proof that if D is a unique factorization domain, then $D[x]$ is also.
N. H. McCoy (Northampton, Mass.).

Neuhaus, Friedrich Wilhelm. Über die Verteilung aller ganzzahligen Gleichungen von mehr als zwei Unbestimmten auf ihre Galois'schen Gruppen. Math. Ann. 121, 379–404 (1950).

The principal results are these. (1) Let $\bar{P}$ be the complex field, $f(x, y, t)$ a polynomial irreducible over $\bar{P}(y, t)$, and $\mathfrak{G}$ its Galois group. Then there is one invariant subgroup, $\mathfrak{H}$, of $\mathfrak{G}$ such that, for all but a finite number of $y\epsilon\bar{P}$, $\mathfrak{H}$ is the Galois group of $f(x, y, t)$ over $\bar{P}(t)$. (2) Let P be the rational field, and $f(x, y, t)$ without affect over $P(y, t)$. Then either the adjunction of an algebraic function of y alone gives a reduction of the Galois group, or $f(x, y, t)$ is without affect over $P(t)$ for all but a finite number of integral y. The paper also contains results on the infrequency of equations with affect which continue previous work of the author [Deutsche Math. 7, 87–116 (1942); these Rev. 8, 248].
G. Whaples (Bloomington, Ind.).

Abstract Algebra

Kneser, Hellmuth. Die komplexen Zahlen und ihre Verallgemeinerung. Math.-Phys. Semesterber. 1, 256–267 (1950).

Expository paper on algebras. *I. Kaplansky.*

Hayashida, Tsuyoshi. Note on Archimedean valuations. Kōdai Math. Sem. Rep., no. 5–6, 7–8 (1949).

The author gives a short proof of the following lemma. If K is the field of real numbers, and if a field $K(\alpha)$ has an Archimedean valuation, then α is algebraic over K. He then shows how this lemma makes possible a much simplified proof of Ostrowski's [Acta Math. 41, 271–284 (1918)] theorem that all Archimedean valuations of an arbitrary field can be obtained by embedding the field in the field of complex numbers. *G. Whaples* (Bloomington, Ind.).

Kuniyoshi, Hideo. On a certain group concerning the p-adic number field. Tôhoku Math. J. (2) 1, 186–193 (1950).

The structure of the group $G(k/K)$ formed by the elements of relative norm 1 in the finite extension K of a p-adic number field k is studied in detail. Let K be an Abelian extension field and $a_{\sigma,\tau}$ a factor set whose exponent is equal to the degree of K/k. It is shown that $G(k/K)$ is generated by $a_{\sigma,\tau}/a_{\tau,\sigma}$ and by the $(1-\sigma)$th powers of the elements of K when σ and τ run through the elements of the Galois group. Next, the structure of the quotient group of $G(k/K)$ with respect to the subgroup generated by these $(1-\sigma)$th powers is given. If $G(k/K)$ coincides with that subgroup, then K/k is cyclic. This, however, is shown not to be true for non-Abelian extensions. A connection between $G(k/K)$ and the maximum Abelian extension field of k is established. It generalizes the known characterization of such Abelian extension fields A of K for which A/k is also Abelian [see Chevalley, J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. 2, 365–476 (1933)]. *O. Todd-Taussky* (Washington, D. C.).

Ancochea, Germán. Concerning the "Nullstellensatz" of Hilbert. Ann. Mat. Pura Appl. (4) 29, 31–34 (1949). (Spanish)

The author's proof of the Nullstellensatz is based on ideas like those used by van der Waerden [Moderne Algebra, vol. 2, 2d ed., Springer, Berlin, 1940, p. 51; these Rev. 2, 120], but the proof is arranged so as to avoid embedding an ideal in a maximal ideal. *I. Kaplansky.*

Specht, Wilhelm. Gesetze in Ringen. I. Math. Z. 52, 557–589 (1950).

This is a rather formal study of the polynomial identities which are satisfied in an associative ring R of characteristic 0. It is shown that in the ring of polynomials with integer coefficients and noncommutative variables, these identities constitute an ideal in which, moreover, replacement of a variable in a polynomial of the ideal by an arbitrary polynomial gives another polynomial of the ideal. The author then proves that all polynomial identities can be derived from those which are multilinear, using the well-known polarization process. Using the theory of linear representations of the symmetric group, he proves moreover that all n-linear polynomial identities in a given number of variables are derived from one of these identities by permutation of the variables and linear combination. When R has a unit element, much more can be said about the multilinear identities; the author proves that they can be derived by the process of forming successive commutators $[x_1, x_2, \cdots, x_n]=[x_1, \cdots, x_{n-1}]x_n - x_n[x_1, \cdots, x_{n-1}]$, and multiplying such commutators in a well determined fashion.
J. Dieudonné (Nancy).

Ballieu, Robert. Anneaux finis à module de type (p, p^2). Ann. Soc. Sci. Bruxelles. Sér. I. 63, 11–23 (1949).

Ballieu, Robert, et Schuind, Marie-Jeanne. Anneaux finis à module de type (p, p^r). Ann. Soc. Sci. Bruxelles. Sér. I. 63, 137–147 (1949).

In the first paper all rings are found whose additive group is of type (p, p^2). The results have been previously announced [same Ann. Sér. I. 61, 222–227 (1947); these Rev. 9, 267].

In the second paper the results are generalized to the case (p, p^r). The method is largely that of case by case classification, meager use being made of structure theory.

I. Kaplansky (Chicago, Ill.).

Raffin, Raymond. Sur les conditions pour qu'un anneau soit à puissances commutatives. C. R. Acad. Sci. Paris **230**, 1488–1489 (1950).

An example is given of a nonassociative ring with commutative principal powers which is not a ring with commutative mixed powers. *R. D. Schafer* (Philadelphia, Pa.).

Nachbin, Leopoldo. On a characterization of the lattice of all ideals of a Boolean ring. Fund. Math. **36**, 137–142 (1949).

Soit S un sup-lattice, c'est-à-dire un ensemble ordonné dans lequel deux éléments quelconques ont une borne supérieure. Un idéal dans S est un ensemble $I\subset S$ tel que $x\epsilon S$, $y\epsilon I$, $x\leqq y$ entraînent $x\epsilon I$ et que sup $(x, y)\epsilon I$ pour deux éléments quelconques x, y de I. L'auteur considère l'ensemble $J(S)$ des idéaux de S ordonné par inclusion; c'est un lattice L qui est caractérisé par les propriétés suivantes: (1) L est complet; (2) tout élément de L est borne supérieure des éléments de L qui lui sont inférieurs et sont compacts dans le sens suivant: x est compact si, lorsqu'il est inférieur à la borne supérieure d'une famille (x_λ), il existe un nombre fini d'indices λ_i tels que x soit inférieur à la borne supérieure des x_{λ_i}. Enfin, l'auteur donne trois conditions supplémentaires pour que L soit un lattice isomorphe à $J(R)$ pour un anneau booléien R, ainsi qu'une sixième condition moyennant laquelle R admet un élément unité.

J. Dieudonné (Nancy).

Nakayama, Tadasi, and Hashimoto, Junji. On a problem of G. Birkhoff. Proc. Amer. Math. Soc. **1**, 141–142 (1950).

The authors give an example to disprove the conjecture that the representation of a finite ordered set A as the product of product-indecomposable factors is essentially unique. They state that this conjecture is true when A is directly sum-indecomposable [see J. Hashimoto, Math. Japonicae **1**, 120–123 (1948); these Rev. **11**, 5] and point out the relationship between the problem under consideration and the uniqueness of factorization in the semi-ring of polynomials in several variables with positive integral coefficients. *L. Nachbin* (Rio de Janeiro).

THEORY OF GROUPS

Engel, Friedrich. Gruppentheorie und Grundlagen der Geometrie. Mitt. Math. Sem. Univ. Giessen **35**, 13 pp. (1945).

Posthumous publication of a manuscript prepared in 1924 as a contribution to a companion volume (never published) to Lobačevskiĭ's Collected Works. It is followed by a letter from Felix Klein commenting on the manuscript.

Piccard, Sophie. Les groupes engendrés par un système connexe de cycles d'ordre sept et les bases des groupes symétrique et alterné de degré $n\geq 10$ dont l'une des substitutions est un cycle du septième ordre. Comment. Math. Helv. **24**, 4–17 (1950).

In this paper the author investigates the subgroups G of the symmetric group S_n generated by various combinations of cycles of order seven for $n=7, 8, 9$ and proves for $n\geqq 10$ that $G=A_n$ if the cycles are connected and permute all of the symbols. Also, if S and T are connected substitutions of S_n for $n>7$ and T is of order seven, a necessary and sufficient condition that they be imprimitive is obtained and if S and T are primitive and $n\geqq 10$, S and T generate S_n if S is odd and A_n if S is even. *G. de B. Robinson.*

Piccard, Sophie. Les classes de substitutions des groupes imprimitifs et les bases des groupes imprimitifs "saturés." Bull. Sci. Math. (2) **73**, 196–214 (1949).

The author has previously shown that it is possible to generalize the familiar division of the elements of the symmetric group S_n into two classes C_1 and C_2 of even and odd substitutions so as to distinguish six classes $C_i, i=1, 2, \cdots, 6$, of substitutions in an imprimitive group G of degree n [C. R. Acad. Sci. Paris **229**, 693–695, 739–741, 1193–1195 (1949); these Rev. **11**, 319]. The author proves here certain relations which hold between these classes; e.g., if G contains one substitution of C_3, then half of the substitutions of G belong to C_3 and the other half to C_4, and the substitutions of C_3 form a normal subgroup of G (similarly for classes C_5 and C_6). If G contains every substitution of S_n which admits a given system of imprimitivity then G is said to be saturated with regard to this system, and G admits no other system of imprimitivity. Moreover, such groups contain elements of C_2, C_4, C_6 and so also of C_1, C_3, C_5, and these latter classes do not overlap, so that G contains three normal subgroups of order $N/2$ if G is of order N. It is also possible to obtain information concerning the number of systems of generators of such groups G. *G. de B. Robinson.*

Tôyama, Hiraku. On commutators of matrices. Kōdai Math. Sem. Rep., no. 5–6, 1–2 (1949).

Soit G un groupe appartenant à l'une des catégories suivantes: groupe unitaire unimodulaire $SU(n)$; groupe unitaire symplectique $USp(n)$; groupe orthogonal propre $O^+(n)$ avec $n>2$. L'auteur démontre que tout élément a d'un tel groupe est de la forme $bcb^{-1}c^{-1}$, autrement dit, est le commutateur de deux éléments convenables du même groupe.

R. Godement (Nancy).

Newton, T. D. A note on the representations of the de Sitter group. Ann. of Math. (2) **51**, 730–733 (1950).

This paper is concerned with the representations of the group of motions of a de Sitter universe by unitary transformations of Hilbert space (there are no finite-dimensional unitary representations of the de Sitter group). A classification of these representations is given which extends and sharpens results obtained previously by L. H. Thomas [Ann. of Math. (2) **42**, 113–126 (1941); these Rev. **2**, 216].

A. Schild (Pittsburgh, Pa.).

Dynkin, E. B. Normed Lie algebras and analytic groups. Uspehi Matem. Nauk (N.S.) **5**, no. 1(35), 135–186 (1950). (Russian)

Une algèbre de Lie L sur un corps valué complet K est dite normée si l'on a introduit sur L une norme $\|x\|$ vérifiant les conditions habituelles relativement à la structure vectorielle de L, avec en outre $\|xoy\|\leqq\|x\|\cdot\|y\|$ pour $x, y\epsilon L$ (on note xoy la multiplication dans L); c'est évidemment le cas si K est le corps des nombres réels ou des nombres complexes. L'auteur montre tout d'abord comment l'on peut associer à

une telle algèbre un noyau de groupe topologique $G(L)$; pour cela, on construit la série formelle qui représente $\log(e^x e^y)$ en fonction de x, y et de leurs commutateurs successifs, et on constate que cette série est absolument convergente lorsque x et y sont en norme assez petits. Ceci permet de définir une multiplication pour des éléments voisins de 0, le produit $x*y$ de deux tels éléments étant égal par définition à la somme de la série en question. L'auteur montre ensuite les relations existant entre homomorphismes, sous-algèbres, idéaux de L d'une part et les notions correspondantes dans $G(L)$ d'autre part. Dans la seconde partie de ce travail, l'auteur rappelle tout d'abord les notions classiques relatifs aux groupes analytiques, puis démontre essentiellement les réciproques des théorèmes établis dans la première partie. En ce qui concerne l'originalité des résultats, le cas "classique" où le corps de base est formé des nombres réels, ou complexes, est bien entendu déja connu; mais la plupart des résultats sont démontrés ici pour les corps p-adiques également. *R. Godement.*

Mackey, George W. Imprimitivité pour les représentations des groupes localement compacts. III. Produits de Kronecker et nombres d'entrelacement forts. C. R. Acad. Sci. Paris **230**, 908–909 (1950).

Cette note fait suite à une précédente note [mêmes C. R. **230**, 808–809 (1950); ces Rev. **11**, 580]. Etant données deux représentations unitaires U et V d'un groupe G, on appelle opérateur d'entrelacement fort de ces représentations tout opérateur d'entrelacement [cf. l'auteur, loc. cit.] T tel que T^*T soit de trace finie; on définit de façon évidente le nombre d'entrelacement fort $J(U, V)$ des deux représentations considérées. À l'aide de cette notion, l'auteur annonce des résultats relatifs à la théorie des représentations induites. Par exemple, soient G_1 et G_2 deux sous-groupes fermés d'un groupe localement compact séparable G, ϑ l'ensemble des classes bilatères G_1xG_2, et ϑ'' l'ensemble des éléments de ϑ de la forme $G_1xy^{-1}G_2$ pour lesquels l'espace homogène $G/(x^{-1}G_1x)\cap(y^{-1}G_2y)$ possède une mesure invariante finie; enfin, soient L et M deux représentations unitaires des sous-groupes G_1 et G_2, U^L et U^M les représentations imprimitives correspondantes de G. Pour toute classe $D\epsilon\vartheta''$, on peut définir $J(L, M, D)$ de façon analogue à $I(L, M, D)$ [cf. l'auteur, loc. cit.]. Supposons alors que la relation d'équivalence sur G définie par les classes bilatères G_1xG_2 vérifie une condition convenable de mesurabilité; notons A l'ensemble des $D\epsilon\vartheta''$ tels que $J(L, M, D)>0$, $m(D)=0$ (m est la mesure de Haar sur G); alors le nombre d'entrelacement fort des représentations imprimitives U^L et U^M est $+\infty$ si la réunion des classes appartenant à A est de mesure >0 sur G; et dans le cas contraire, ce nombre est égal à la somme des $J(L, M, D)$ où D parcourt l'ensemble des classes appartenant à ϑ'' qui ne sont pas de mesure nulle sur G. La définition de l'ensemble A et le théorème sur le nombre d'entrelacement fort sont corrigés suivant une communication de l'auteur. *R. Godement* (Nancy).

Vilenkin, N. Ya. Investigations on the theory of topological Abelian groups. Uspehi Matem. Nauk (N.S.) **5**, no. 2(36), 208–210 (1950). (Russian)

This is an abstract of the author's doctoral dissertation. According to the abstract the results in this dissertation have appeared in other articles of the author [Rec. Math. [Mat. Sbornik] N.S. **19**(61), 311–340 (1946); Mat. Sbornik N.S. **22**(64), 135–177 (1948); **24**(66), 189–226 (1949); Doklady Akad. Nauk SSSR (N.S.) **61**, 969–971 (1948); **65**, 3–5 (1949); these Rev. **8**, 312; **9**, 497; **10**, 679, 282, 507]. *G. W. Mackey* (Cambridge, Mass.).

Vikhrov, A. Theory of extensions of ultragroups. Učenye Zapiski Moskov. Gos. Univ. **100**, Matematika, Tom I, 3–19 (1946). (Russian. English summary)

Soit H un hypergroupe, ayant une unité bilatère. Un sous-hypergroupe h de H est dit fortement invariant dans H si, pour tout $m\epsilon H$, il existe des m', $m''\epsilon H$ tels que $m'hm\subseteq h$ et $mhm''\subseteq h$. Dresher et Ore ont prouvé que, dans ce cas (et dans ce cas seulement), H/h est un groupe. Etant donné deux groupes G et g, l'auteur pose, pour les hypergroupes, un problème analogue à celui de Schreier pour les groupes. Trouver tous les hypergroupes $H\supset g$, tels que H/g soit $\simeq G$ (ce qui entraîne que g est fortement invariant dans H). Il impose, toutefois, à H des réstrictions supplémentaires: H possède une unité bilatère et tout $a\epsilon g$ est un scalaire bilatère de H. La solution de ce problème est assez analogue à celle du problème original du Schreier. Les extensions nonéquivalents (au même sens que chez Schreier) H de g par G correspondent biunivoquement aux classes d'équivalence de systèmes de facteurs, constitués comme suit. (1) À tout $\alpha\epsilon G$ est attaché un homomorphisme φ_α de g sur lui-même, dont le noyau soit g_α. En particulier, si ϵ est l'unité de G, et si e est celle de g, on a $g_\epsilon=\{e\}$. (2) À tout couple $\alpha, \beta\epsilon G$ est attachée une réunion $u_{\alpha,\beta}$ de classes (mod $g_{\alpha\beta}$) dans g, satisfaisant, pour tout $a\epsilon g$, aux conditions suivantes:

(a) $u_{\epsilon,\epsilon}^{-1}au_{\epsilon,\epsilon}=a^{\varphi_\epsilon}$, (b) $u_{\alpha,\beta\gamma}a^{\varphi_{\beta\gamma}}u_{\beta,\gamma}=u_{\alpha\beta,\gamma}(u_{\alpha,\beta}a^{\varphi_\beta})^{\varphi_\gamma}$.

Deux systèmes de facteurs $(\varphi_\alpha, u_{\alpha,\beta})$ et $(\varphi'_\alpha, u'_{\alpha,\beta})$ sont équivalents s'il existe une application $\alpha\rightarrow c_\alpha$ de G dans g telle que $\varphi'_\alpha=\varphi_\alpha(c_\alpha)$ (où (c_α) est l'automorphisme intérieur $a\rightarrow c_\alpha a c_\alpha^{-1}$ de g) et $u'_{\alpha,\beta}=c_{\alpha\beta}^{-1}u_{\alpha,\beta}c_\alpha^{\varphi_\beta}c_\beta$. La méthode de l'auteur est semblable à celle de Schreier. Un problème analogue, mais avec d'autres hypothèses sur H (on supposait que les $a\epsilon g$ sont scalaires à droite, mais non forcement à gauche, dans H, mais on leur imposait certaines autres conditions), a été résolu par le reférant [C. R. Acad. Sci. Paris **218**, 542–544 (1944); ces Rev. **6**, 202]. *M. Krasner.*

NUMBER THEORY

Thébault, Victor. Sur des nombres curieux. Ann. Soc. Sci. Bruxelles. Sér. I. **62**, 101–108 (1948).

Thébault, Victor. Sur les nombres premiers impairs et sur certaines puissances des nombres entiers consécutifs. Mathesis **59**, 10–12 (1950).

Pekelharing, N. R. The number 41. Simon Stevin **27**, 93–98 (1950). (Dutch)

Some remarks on Euler's formula n^2+n+41, which represents primes for $n=0, 1, 2, \cdots, 39$. *N. G. de Bruijn.*

Beumer, M. G. On factorization of prime numbers of the form $4t+1$. Nieuw Tijdschr. Wiskunde **37**, 349–351 (1950). (Dutch)

Venkataraman, C. S. A generalisation of Euler's ϕ function. Math. Student **17** (1949), 34–36 (1950).

If g is a divisor of M, the number of integers x $(0<x\leqq M)$ with $(g, M)=1$ is denoted by $\phi(M, g)$. The author seems to be unaware of the triviality of the relation $\phi(M, g)=\phi(M/g)$. *N. G. de Bruijn* (Delft).

Bernhard, H. A. On the least possible odd perfect number. Amer. Math. Monthly 56, 628–629 (1949).

The author points out that no odd perfect number exists which is less than 10^{11}. [See the following review.]

A. Brauer (Chapel Hill, N. C.).

Kühnel, Ullrich. Verschärfung der notwendigen Bedingungen für die Existenz von ungeraden vollkommenen Zahlen. Math. Z. 52, 202–211 (1949).

Sylvester [C. R. Acad. Sci. Paris 106, 522–526 (1888)] proved that an odd perfect number N must have at least 5 different prime divisors. Moreover, he stated that he was able to prove that at least 6 different primes must divide N. The author proves this result of Sylvester. He concludes that no odd perfect number exists which is less than $2.2\cdot 10^{12}$.

A. Brauer (Chapel Hill, N. C.).

Kühnel, Ullrich. Über die Anzahl der Produktdarstellungen der positiven ganzen Zahlen. Arch. Math. 2, 216–219 (1950).

Let $f(n)$ be the number of different representations of n as a product of factors greater than 1. Two representations which differ only by the order of the factors are considered as different. The function $f(n)$ was studied by Kálmár [Acta Litt. Sci. Szeged 5, 95–107 (1931)] and Hille [Acta Arith. 2, 134–144 (1936)]. The author states some new formulas for $f(n)$ without proof, the proofs having been given in the author's unpublished dissertation. From his formulas the author obtained some relations for binomial coefficients.

A. Brauer (Chapel Hill, N. C.).

Gupta, Hansraj. On a problem of Erdös. Amer. Math. Monthly 57, 326–329 (1950).

P. Erdös stated and J. Lampek proved [same Monthly 55, 103 (1948)] that for every k, the equation (*) $\varphi(x)=k!$ is solvable, φ being Euler's totient function. V. L. Klee conjectured [same Monthly 56, 21–22 (1949)] that $S_1(k)\geqq 1$ for every k, and that $S_1(k)$ tends to infinity with k; $S_1(k)$ denotes the number of solutions of (*) where x is such that in its canonical factorisation just one prime appears to the first power. These conjectures are proved here.

N. G. de Bruijn (Delft).

van der Corput, J. G. On de Polignac's conjecture. Simon Stevin 27, 99–105 (1950). (Dutch)

Let $N(x)$ denote the number of odd positive integers $n\leqq x$ which are not of the form $n=2^k+p$ (p prime). De Polignac conjectured [Nouv. Ann. Math. Paris (1) 8, 423–429 (1849)] that $N(x)=0$ for all x (counting 1 as a prime), but Euler already knew the examples $n=127$ and $n=959$. It is proved here that $N(x)/x$ has a positive lower limit, as $x\rightarrow\infty$. On the other hand Romanoff proved [Math. Ann. 109, 668–678 (1934)] that the upper limit is less than 1.

N. G. de Bruijn (Delft).

Eljoseph (Kabaker), Nathan. On products of integers. Riveon Lematematika 3, 60–64 (1949). (Hebrew. English summary)

The author gives a new proof of the generalized Wilson theorem. His methods also yield results on the number of quadratic residues mod m and the value (mod m) of their product.

A. Dvoretzky (New York, N. Y.).

Jarden, Dov. Remark to Sierpinski's theorem on isolated primes. Riveon Lematematika 3, 68–69 (1949). (Hebrew. English summary)

The reference is to W. Sierpiński, Colloquium Math. 1, 193–194 (1948) [these Rev. 10, 431]. *A. Dvoretzky.*

Gatteschi, L., e Rosati, L. A. Risposta ad una questione proposta da A. Moessner. Boll. Un. Mat. Ital. (3) 5, 43–48 (1950).

In reply to a question proposed by A. Moessner [same Boll. (3) 4, 146 (1949)] the authors show that the Diophantine system

$$A_1^2+B_1^2=A_2^2+B_2^2=A_3^2+B_3^2,\quad A_1^3+B_1^3=A_2^3+B_2^3=A_3^3+B_3^3$$

has no real solutions in which the pairs (A_1, B_1), (A_2, B_2), (A_3, B_3) are distinct. Moreover, the Diophantine system $A_1^2+B_1^2=A_2^2+B_2^2$, $A_1^3+B_1^3=A_2^3+B_2^3$ has no nontrivial integer solutions. *W. H. Simons* (Kingston, Ont.).

Mordell, L. J. Note on cubic equations in three variables with an infinity of integer solutions. Ann. Mat. Pura Appl. (4) 29, 301–305 (1949).

The equation $z^3=ax^3+by^3+c$ has an infinite number of integral solutions in x, y, z if a, b, c are odd, $(a, b)=1$, $c\not\equiv a^3$ (mod 7) when $7|b$, and $c\not\equiv b^3$ (mod 7) when $7|a$. The proof is based on the lemma that $y^3\equiv x^2+k$ (mod a) is solvable if a is odd; in fact it is solvable with $(y, a)=1$ except when $7|a$ and $7|(k+1)$. *I. Niven* (Eugene, Ore.).

Gloden, A. Une méthode de résolution du système trigrade normal. Bull. Soc. Roy. Sci. Liége 18, 516–518 (1949).

A solution in integers of the system of equations

$$\sum_{i=1}^{s} x_i^k=\sum_{i=1}^{s} y_i^k,\qquad k=1, \cdots, m,$$

is called normal (or ideal) when $s=m+1$. Methods for obtaining all solutions of the normal trigrade equation $\sum_{i=1}^4 x_i^k=\sum_{i=1}^4 y_i^k$, $k=1, 2, 3$, have been given by Dickson [Introduction to the Theory of Numbers, University of Chicago Press, 1929, pp. 49–52] and Buquet [Sciences, Revue de l'Association Française pour l'Avancement des Sciences 75, no. 60, 456 (1948)]. In the present paper the author gives another method which results in a complete solution in terms of 8 arbitrary parameters.

W. Simons (Vancouver, B. C.).

Nagell, Trygve. Über die Darstellung ganzer Zahlen durch eine indefinite binäre quadratische Form. Arch. Math. 2, 161–165 (1950).

From a solution (u, v) of (1) $u^2-Dv^2=N$ an associated solution $u'=ux+vyD$, $v'=uy+vx$ can be obtained by using a solution (x, y) of (2) $x^2-Dy^2=1$, where D is a positive nonsquare integer. A complete set of associated solutions of (1) is an equivalence class; define the fundamental solution of a class as that one with least positive v and the corresponding u (the positive one in case of ambiguity). It is proved that the fundamental solution of a class satisfies the inequality $v\leqq\frac{1}{2}y_1\{2|N|/(x_1\pm 1)\}^{\frac{1}{2}}$, with sign chosen to match that of N, and where (x_1, y_1) is a fundamental solution of (2). Thus (1) has only a finite number of classes. If N is a prime p, then (1) has at most one nonnegative solution satisfying this inequality on v, and (assuming the existence of a solution) there are one or two classes according as $p|2D$

or not. It should be pointed out that inequalities for the least positive solution of $x^2-Dy^2=4$, $D\equiv 0$ or 1 (mod 4), have been given by L. K. Hua [Bull. Amer. Math. Soc. **48**, 731–735 (1942); these Rev. **4**, 130]. Earlier work was done by I. Schur. *I. Niven* (Eugene, Ore.).

Dueball, Fritz. Bestimmung von Polynomen aus ihren Werten mod p^n. Math. Nachr. **3**, 71–76 (1949).

Let p be a prime and n a fixed positive integer. Let $f(x)$ be a polynomial with rational integral coefficients. If $f(x)$ is given for $x=0, 1, \cdots, p^n-1$, then $f(x)$ is determined mod p^n for all integral values of x. On the other hand, when $n>1$, the values of $f(x)$ cannot be arbitrarily prescribed for a complete set of residues mod p^n, but are already determined when x runs through a certain subset of residues. In fact, the author proves that the values of $f(x)$ are determined mod p^n for all integral values of x, if $f(x)$ is known for $x=0, 1, \cdots, (\alpha+1)p-1$, where α is defined as follows: let p^{n_ν} be the highest power of p which divides $p^\nu\nu!$. The integers $n_0\ (=1), n_1, \cdots$ form a monotonic increasing sequence and hence, corresponding to every n, there exists a unique integer $\alpha=\alpha(n)$ such that $n_\alpha<n\leqq n_{\alpha+1}$. A simple procedure is developed for constructing polynomials that take prescribed values subject to the restrictions referred to. Furthermore, a formula is derived giving the most general polynomial which vanishes mod p^n for all integral values of x.
W. Ledermann (Manchester).

Schwarz, Štefan. On universal forms in finite fields. Časopis Pěst. Mat. Fys. **75**, 45–50 (1950). (English. Czech summary)

The following theorem is proved. Let (1) $a_1, \cdots, a_k \epsilon GF(p^n)$, $a_1\cdots a_k\neq 0$; (2) $(p^n-1, k)\leqq p-1$. Then the equation $b=a_1x_1^k+\cdots+a_kx_k^k$ has a solution with $x_i\epsilon GF(p^n)$ for every $b\epsilon GF(p^n)$. The author had previously proved a like result with (2) replaced by $k\,|\,(p-1)$ [Quart. J. Math., Oxford Ser. (1) **19**, 160–163 (1948); these Rev. **10**, 101]. *L. Carlitz.*

Martino, Caio Manlio. Equazioni relative a poligoni regolari ordinari e stellati, di qualunque numero di lati. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **11(80)** (1947), 227–235 (1949).

If the unit circle is divided into n equal parts and if one of the points of division is joined to all the others one obtains $n-1$ chords having the $[n/2]$ different lengths: $2\sin \pi k/n$, $k=1, 2, \cdots, [n/2]$. The author proposes to write the equation whose roots are these different lengths but reformulates the problem so as to require expressions for the coefficients of the polynomial

$$F_n(x)=\prod_{k=1}^{[(n-1)/2]}(x^2-4\sin^2 k\pi/n).$$

It is stated that

$$F_n(x)=\begin{cases}\displaystyle\sum_{k=0}^{\infty}(-1)^k\binom{n-k-1}{k}x^{n-2k-1}, & \text{if } n \text{ is even,}\\ \displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{n}{k}\binom{n-k-1}{k-1}x^{n-2k-1}, & \text{if } n \text{ is odd;}\end{cases}$$

$F_n(x)$ is given explicitly for $n=3(1)8$, 20, 100. The connection between $F_n(x)$ and Lucas' U_n and V_n is not indicated [E. Lucas, Théorie des nombres, Paris, 1891, pp. 312–314].
D. H. Lehmer (Berkeley, Calif.).

Jacobsthal, Ernst. Über absolut multiplikative zahlentheoretische Funktionen. I. Norske Vid. Selsk. Forh., Trondheim **22**, no. 12, 42–46 (1950).

Jacobsthal, Ernst. Über absolut multiplikative zahlentheoretische Funktionen. II. Norske Vid. Selsk. Forh., Trondheim **22**, no. 13, 47–50 (1950).

Congruence properties of the symmetric functions of the numbers $a_1, \cdots, a_t$ less than n and prime to n can be obtained by Bauer's theorem on the elementary symmetric functions [cf. Hardy and Wright, Theory of Numbers, 2d ed., Oxford, 1945, chapter 8]. In this paper it is shown that it is sometimes advantageous to use power sums $s_r=\sum a_\lambda^r$ instead. The author gives an explicit formula for s_r, using the Möbius function and Bernouillian coefficients. The result is applied to a proof of Leudesdorff's theorem on $\sum a_\lambda^{-1}$. Further, by Newton's formulas, the elementary symmetric functions $f_r=\sum a_1a_2\cdots a_r$ can be expressed in terms of $s_1, \cdots, s_r$. The expressions obtained for s_1, s_2, s_3, s_4 and for f_1, f_2, f_3, f_4 are stated explicitly. *N. G. de Bruijn.*

Jacobsthal, Ernst. Über einen Satz von Leudesdorf. Norske Vid. Selsk. Forh., Trondheim **22**, no. 41, 193–197 (1950).

Let $S_1=\sum x_\nu^{-1}$, where $1\leqq x_\nu\leqq n$, $(x_\nu, n)=1$. By the theorem of Leudesdorff

$$S_1\equiv\begin{cases}0 \pmod{n^2/4}, & n=2^e,\\ 0 \pmod{n^2/6}, & n\neq 2^e,\ d=(n,6).\end{cases}$$

In the present paper the residue of S_1 (mod n^2) is obtained. As a corollary and making use of a result of S. Selberg [~~forthcoming paper~~], it is proved that $S_1\equiv 0$ (mod n^2) for almost all n. *L. Carlitz* (Durham, N. C.).
[Same Forh. 23, 1-2 (1951); these Rev. 13, 15]

Kruyswijk, D. On some well-known properties of the partition function $p(n)$ and Euler's infinite product. Nieuw Arch. Wiskunde (2) **23**, 97–107 (1950).

Let $f(x)=\prod_{n=1}^{\infty}(1-x^n)$ denote Euler's product and let $A_m=A_m(x)=x^mf(x^{24m})$. The author gives proofs, involving only the calculus of formal power series, of the following Ramanujan identities

$$5f^5(x^5)f^{-6}(x)=\sum_{r=0}^{\infty}p(5r+4)x^r,$$

$$7f^3(x^7)f^{-4}(x)+49xf^7(x^7)f^{-8}(x)=\sum_{r=0}^{\infty}p(7r+5)x^r,$$

$$A^4_1+5A^3_1A_{25}+15A^2_1A^2_{25}+25A_1A^3_{25}+25A^4_{25}=A^6_5(A_1A_{25})^{-1},$$

$$A^8_1+7A^5_1A_{49}+21A^4_1A^2_{49}+49A^3_1A^3_{49}+7A^3_1\{A^4_7+21A^4_{49}\}$$
$$+7A_1\{5A^4_7+49A^4_{49}\}A_{49}+49A^4_7A^2_{49}+343A^6_{49}=A^8_7(A_1A_{49})^{-1},$$

where $p(n)$ denotes the number of unrestricted partitions of n. The paper closes with the author's presentation of Jacobi's proof of the triangular number theorem which he writes in the form $f^3(x)=\sum_{k=0}^{\infty}(-1/k)kx^{(k^2-1)/8}$.
D. H. Lehmer (Berkeley, Calif.).

Bellman, Richard. Ramanujan sums and the average value of arithmetic functions. Duke Math. J. **17**, 159–168 (1950).

By a method which was outlined in a preceding paper [Proc. Nat. Acad. Sci. U. S. A. **34**, 149–152 (1948); these Rev. **9**, 499] the author proves that, for $\Re(s)>0$, $N\to\infty$, we have

$$(1)\qquad \sum_{n\leqq N}\sigma_{-s}(f(n))\sim c_1(s)N,$$

$$(2)\qquad \sum_{p\leqq N}\sigma_{-s}(f(p))\sim c_2(s)N/\log N.$$

Without proof, he states that

$$(3) \qquad \sum_{n \leq N} \sigma_{-s}(f(n))^r \sim C_r(s)N, \qquad r=1, 2, 3, \cdots.$$

Here $\sigma_{-s} = \sum_{k|n} k^{-s}$, and $f(x)$ is a polynomial with integer coefficients (the theorems are stated for integer-valued polynomials, but the proofs do not consider this generalization). The functions $c_4(s)$ and $c_6(s)$ are, respectively,

$$c_4(s) = \zeta(1+s)\sum_{n=1}^{\infty} \rho(n)n^{-1-s}; \quad c_6(s) = \zeta(1+s)\sum_{n=1}^{\infty} \rho'(n)n^{-s}/\phi(n).$$

Both series converge absolutely for $\Re(s)>0$. Here $\rho(n)$ denotes the number of solutions of $f(x)\equiv 0 \pmod n$ if x runs through a complete set of residues mod n; $\rho'(n)$ is the corresponding number for a reduced set. In formula (2.3), $\mu(n/d)$ should be $\mu(q/d)$; this leads to the error term $O(d(n))$ instead of $O(d(n) \log Y)$. *N. G. de Bruijn* (Delft).

Heilbronn, H. On Euclid's algorithm in cubic self-conjugate fields. Proc. Cambridge Philos. Soc. **46**, 377–382 (1950).

It is proved in this paper that Euclid's algorithm can hold in only a finite number of cyclic cubic fields. It is first shown, by consideration of the class-number, that it suffices to prove the result when the discriminant of the field is d^2, where d is a large prime $\equiv 1 \pmod 6$. Now let $\chi(n)$ denote a cubic character (mod d). All numbers prime to d fall into three classes A, B, C according as $\chi(n)=1$, ω, or ω^2, where ω is a complex cube root of unity. The main lemma for the proof is that d is representable as $b_1c_1+b_2c_2$, where b_1, b_2 are positive integers in B, and c_1, c_2 are positive integers in C, and $(b_1, c_1)=(b_2, c_2)=1$. The proof of this requires the consideration of several cases, and the use of estimates for certain character-sums. *H. Davenport* (London).

Cohn, Harvey. Minkowski's conjectures on critical lattices in the metric $(|\xi|^p+|\eta|^p)^{1/p}$. Ann. of Math. (2) **51**, 734–738 (1950).

Continuing work started by C. S. Davis [J. London Math. Soc. **23**, 172–175 (1948); these Rev. **10**, 512, 856], the author shows that Minkowski's conjecture about the critical lattices of the convex region $|\xi|^p+|\eta|^p \leq 1$ where $p \geq 1$ [Minkowski, Diophantische Approximationen, 1st ed., Teubner, Leipzig, 1907, pp. 51–58] is true for sufficiently large positive p, but false in certain small finite intervals for p. *K. Mahler.*

ANALYSIS

Calculus

Karamata, J. Sur certains cas particuliers du premier théorème de la moyenne. Bull. Soc. Math. Phys. Serbie **1**, no. 3–4, 83–103 (1949). (Serbian. French summary)

Bojanić, R. Une propriété caractéristique des courbes du second degré. Bull. Soc. Math. Phys. Serbie **1**, no. 3–4, 105–111 (1949). (Serbian. French summary)

For a given $F(x)$ the number ξ satisfying

$$(1) \qquad F(y)-F(x)=(y-x)F'(\xi)$$

is a function of x and y; Karamata asks for functions F and φ such that the ξ corresponding to F satisfies (2) $\varphi(\xi)=\frac{1}{2}\{\varphi(x)+\varphi(y)\}$. He finds that a solution is possible if and only if $\varphi'(\xi)=(Ax^2+Bx+C)^{-\frac{1}{2}}$ and then $F''(x)=D\varphi'^3(x)$. He also considers a more general requirement on ξ of the form $\xi(x, y)=\int_0^1 \varphi^{-1}\{\varphi(x)p(t)+\varphi(y)q(t)\}dt$, φ monotonic and $p+q=1$, $p(t)=q(1-t)$, and determines the corresponding solutions φ.

In the second paper Bojanić points out that the solutions $F(x)$ of (1) and (2) are quadratic and hence that (1) and (2) determine the conic sections. He then gives a geometric interpretation of (1) and (2) as properties of conic sections. *R. P. Boas, Jr.* (Evanston, Ill.).

Ważewski, Tadeuz. Une démonstration uniforme du théorème généralisé de L'Hôspital. Ann. Soc. Polon. Math. **22** (1949), 161–168 (1950).

Let real functions f, g be defined on a finite or infinite open interval with a finite or infinite extremity k. Let l and λ denote the inferior limits of $f(x)/g(x)$ and $f'(x)/g'(x)$, respectively, as $x \to k$; let L and Λ denote the corresponding superior limits. The generalized theorem of L'Hôspital states that, under suitable conditions, $\lambda \leq l \leq L \leq \Lambda$. The author proves this theorem with the aid of arguments very similar to those used by him in an earlier paper [Prace Mat.-Fiz. **47**, 117–128 (1949); these Rev. **11**, 585]. *A. E. Taylor* (Los Angeles, Calif.).

de La Vallée Poussin, Ch. J. Sur la différentielle totale. Ann. Soc. Sci. Bruxelles. Sér. I. **64**, 74–75 (1950).

By considering the expression $F_1(x, y)=\int_b^y F(x, y)dy$, this note proves that a necessary condition that

$$P(x, y)dx+Q(x, y)dy,$$

P and Q continuous, be the total differential of the function $F(x, y)$ is that

$$\int_a^x P(x, y)dx+\int Q(a, y)dy=\int_b^y Q(x, y)dy+\int P(x, b)dx,$$

which is obviously necessary. *T. H. Hildebrandt.*

Kloosterman, H. D. Derivatives and finite differences. Duke Math. J. **17**, 169–186 (1950).

Let k and r be two positive integers, and h a real number $\neq 0$. Let $f(x)$ be a real function defined in the interval I: $[x, x+(k+r-1)h]$. We denote by $\Delta_h^n f(x)$, $n=1, 2, \cdots, k+r-1$, the nth difference of $f(x)$ with increment h, and by $Q_n(r)$ the coefficient of x^n in the expansion of $([\log(1+x)]/x)^r$. In his first main theorem the author proves that, if $f(x)$ has $k+r$ derivatives such that $f^{(k+r)}(x)$ is bounded, then there exists a number ξ in I, such that

$$f^{(r)}(x)=\sum_{\nu=0}^{k-1} Q_\nu(r)h^{-r}\Delta_h^{r+\nu}f(x)+Q_k(r)h^k f^{(k+r)}(\xi).$$

An analogous theorem (the second main theorem) is proved for functions of a discrete variable. The use of both theorems is illustrated by giving some simple proofs of known results. Thus, for example, by means of the first main theorem (or rather by a less precise but easier-to-prove version of it) a simplified proof of a theorem of Boas and Pólya [Duke Math. J. **9**, 406–424 (1942); these Rev. **3**, 292] is indicated. *S. Agmon* (Houston, Tex.).

DeHeer, W. J. C. A mortality formula which has practically the same advantages as that of Makeham. Verzekerings-Arch. 28, 201–210 (1950). (Dutch)

Du Motel [Bull. Inst. Actuar. Français 7, 73–77 (1896)] proved that if the following relation holds for all real $t>0$,

$$\prod_{j=1}^{m}\{l(x_j)-l(x_j+t)\}/l(x_j)=[\{l(w)-l(w+t)\}/l(w)]^m,$$

where $l(x)$ is monotonically decreasing $(x>0)$ and thrice differentiable, and m is, by implication, a positive integer, then this implies that $l''(x)/l'(x)$ is independent of x. The author embellishes du Motel's proof and provides a new and simpler one of his own. *H. L. Seal* (New York, N. Y.).

Parker, S. T. Convergence factor and regularity theorems for convergent integrals. Duke Math. J. 17, 91–110 (1950).

The first part of the paper is concerned with conditions on functions $\varphi(x, y)$, in a specified class C, which are necessary and sufficient to ensure

$$\lim_{\alpha\to\alpha_0}\lim_{x\to\infty}\int_0^x\varphi(t,\alpha)f(t)dt=\lim_{x\to\infty}\int_0^x f(t)dt$$

whenever the right member exists. The conditions explicitly (and implicitly, to justify use of integration by parts) imposed on the functions in class C are so severe that simple necessary and sufficient conditions are easily obtained. The author's proof of necessity of the condition (*) $\int_0^\infty|\varphi_t'(t,\alpha)|dt<K(\alpha)$ is superfluous, and this condition can be omitted from the list of necessary and sufficient conditions because each $\varphi(t,\alpha)$ in C is, for each α, a monotone bounded function of t automatically satisfying (*). The major and more involved part of the paper deals with the analogous problem obtained by replacing the single variable t by two or more variables. *R. P. Agnew.*

Erugin, N. P., and Sobolev, S. L. Approximate integration of some oscillating functions. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 193–196 (1950). (Russian)

Let $f(x)$ satisfy $|f^{(m)}(x)-f^{(m)}(x+h)|\leqq K_m|h|$, and let $\phi(x)$ be periodic with period k. A method for evaluation of the integral $J=\int_0^{nk}f(x)\phi(x)dx$ is contained in this paper. With the auxiliary function $\Phi_m(x)=\int_0^x\Phi_{m-1}(t)dt+k^{-1}\int_0^k t\Phi_{m-1}(t)dt$; $\Phi_0(x)=\phi(x)$, the authors find that if $\int_0^k\phi(x)dx=0$,

$$J=\sum_{l=0}^{m-1}[f^{(l)}(nk)-f^{(l)}(0)](-1)^l k^{-1}\int_0^k t\Phi_l(t)dt+(-1)^m J_m,$$

where $|J_m|\leqq 4MK_m n(k/4)^{m+2}$, $M\geqq|\phi(x)|$. Cases for which $\int_0^k\phi(x)dx\neq 0$ are also considered. *R. E. Gaskell.*

***Gans, Richard. Vektoranalysis mit Anwendungen auf Physik und Technik.** 7th ed. B. G. Teubner, Leipzig, 1950. 120 pp. $1.40.

This is a revision by W. Stein of the 6th edition [Teubner, Leipzig, 1929]. The general plan and contents are unaltered but increased attention has been given to direct proofs. Contents: (I) Elementary operations of vector analysis, (II) Differential operations, (III) Curvilinear coordinates, resolution, mechanical deformation, (IV) Symmetric tensors, (V) Applications to hydrodynamics and electrodynamics.

L. M. Milne-Thomson (Greenwich).

Boggio, Tommaso. Sopra due notevoli formule di calcolo vettoriale. Boll. Un. Mat. Ital. (3) 5, 54–56 (1950).

Proofs of the distributive law for vector multiplication and the triple vector product. *L. M. Milne-Thomson.*

Emde, Fritz. Der Einfluss der Feldlinien auf Divergenz und Rotor. Arch. Elektrotechnik 39, 2–8 (1948).

Geometrical interpretations of div $\mathbf{t}$ and curl $\mathbf{t}$, where $\mathbf{t}$ is a unit field vector. *L. M. Milne-Thomson* (Greenwich).

Theory of Sets, Theory of Functions of Real Variables

Leja, F. Une généralisation de l'écart et du diamètre transfini d'un ensemble. Ann. Soc. Polon. Math. 22 (1949), 35–42 (1950).

Let $\phi(p_1, p_2, \cdots, p_\alpha)$ be a positive, continuous and symmetric function of α points in a metric space and let $\phi=0$ if two of these points coincide. Let E be a closed set, p_i arbitrary on E, and $V(p_1,\cdots,p_n)=\prod\phi(p_{i_1},\cdots,p_{i_\alpha})$, where $i_1,\cdots,i_\alpha$ runs over all possible combinations of $1, 2, \cdots, n$ taken α at a time; hence this product has $\binom{n}{\alpha}$ factors. Let $V_n=\max V$ where the p_i move arbitrarily on E. The author proves that $\lim_{n\to\infty}\binom{n}{\alpha}^{-1}\log V_n$ exists. Consider on the other hand $\Delta_k(p_1,\cdots,p_n)=\prod\phi(p_k, p_{i_2},\cdots,p_{i_\alpha})$ where $i_2, i_3, \cdots, i_\alpha$ runs over all possible combinations of $1, 2, \cdots, k-1, k+1, \cdots, n$ taken $\alpha-i$ at a time. Let

$$\Delta_n=\sup_{p_i\subset E}\{\min_k\Delta_k(p_1, p_2,\cdots,p_n)\}.$$

Then $\lim\binom{n-1}{\alpha-1}^{-1}\log\Delta_n$ exists and coincides with the limit obtained before. *G. Szegö* (Stanford University, Calif.).

Łoś, J., and Marczewski, E. Extensions of measure. Fund. Math. 36, 267–276 (1949).

Let μ be a finitely additive measure defined on a field **M** of elements of a Boolean algebra **A**. The main result of this paper is the following. To every number ξ between the inner and outer μ-measure of an arbitrary element Z of **A** there exists an extension ν of μ to the field generated by **M** and Z such that $\nu(Z)=\xi$. The method of extension used is effective only if the outer μ-measure of Z is finite. The paper ends with both a discussion of conditions under which this method is unique and an example showing that the above result is not true for all countably additive measures in σ-fields. *H. M. Schaerf* (St. Louis, Mo.).

Minetti, Silvio. Sull'operazione di derivazione. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 27–31 (1950).

Let I be a set of nonnegative numbers with 0 as an accumulation point and let f be a real-valued function defined on I. The author defines the right oscillatory derivative of f at 0 as $\lim_{x\to 0}(\sup_{0\leq t\leq x}f(t)-\inf_{0\leq t\leq x}f(t))$, where t and x lie in I. Theorem: If $0\epsilon I$ and if the right oscillatory derivative of f at 0 is 0, then the ordinary right derivative of f at 0 exists and is 0. Two examples, $f_1=$ the characteristic function of the irrationals and $f_2(x)=x\sin 1/x$, show that the right oscillatory derivatives of f_i $(=i)$ can be finite even when the ordinary right derivative fails to exist.

M. M. Day (Urbana, Ill.).

Zahorski, Zygmunt. Supplément au mémoire "Sur l'ensemble des points singuliers d'une fonction d'une variable réelle admettant les dérivées de tous les ordres." Fund. Math. **36**, 319–320 (1949).

In the review of the cited memoir [Fund. Math. **34**, 183–245 (1947); these Rev. **10**, 23] it was stated that a theorem of Pringsheim, proved by the author, had previously been proved by V. Ganapathy Iyer and by R. P. Boas [see the cited review for a statement of the theorem and further references]. The author points out in this supplement that Ganapathy Iyer had proved only a weaker theorem.

Rodnyanskiĭ, A. M. On differentiable mappings of regions. Doklady Akad. Nauk SSSR (N.S.) **72**, 15–17 (1950). (Russian)

Let G be a region in R^n, $n \geqq 2$. Let $f_1, f_2, \cdots, f_n$ be real-valued functions with domain G and possessing total differentials throughout G. Let f be the mapping of G into R^n defined by $f_1, \cdots, f_n$, and let J denote the Jacobian of f. Let G^+, G^-, and G^0 denote the subsets of G where J is >0, <0, and $=0$, respectively. For $E \subset R^n$, let E_g denote the set $E^- \cap E'^-$, and let mes E denote the measure of the set E. The following assertions are made. (1) If $G^+ \neq 0$ and $G^- \neq 0$, then $G^0 \neq 0$. (2) If $G^0 = 0$, then for every $a \epsilon G$, there exist connected neighborhoods $U(a)$ and $V(f(a))$ such that f maps $U(a)$ homeomorphically onto $V(f(a))$. (3) If $a \epsilon G$ and $J(a) \neq 0$, then there exists an arbitrarily small connected neighborhood $U(a)$ such that $f(U(a))$ is a region. (4) $G = G^0$ if and only if $f_1, \cdots, f_n$ are functionally dependent in G. (5) mes $fG^0 = 0$. If $G^+ \neq 0$, then mes $G^+ > 0$ and mes $fG^+ > 0$. If mes $fG > 0$, then $(fG)'^{-'} \neq 0$. (6) Suppose that G is bounded and that f admits a continuous extension over G^-. Then if $(fG)_g \subset f(G_g)$, the functions $f_1, \cdots, f_n$ have maximum and minimum values on G^- which are assumed on G_g. A number of other results are announced; no proofs are given. *E. Hewitt* (Seattle, Wash.).

Mickle, E. J., and Radó, T. On upper semicontinuous functions. Proc. Amer. Math. Soc. **1**, 226–230 (1950).

Let X be a metric space, let Y be a compact metric space and let $F(x, y)$ be a real-valued, bounded, upper-semicontinuous function defined on the product space $X \times Y$. The main result of the authors is that there exists a Borel transformation $T: y = \Phi(x)$ from X into Y such that for each $x \epsilon X$ we have $F[x, \Phi(x)] = M(x)$, where $M(x) = \max F(x, y)$ for x fixed, $y \epsilon Y$. Here, by a Borel transformation T is meant a single-valued transformation such that for every closed set $F \subset Y$ the inverse set $T^{-1}(F)$ is a Borel set in X [Kuratowski, Topologie I, 1st ed., Warsawa-Lwów, 1933]. The above statement generalizes, using new reasoning, an analogous statement needed in Lebesgue area theory, where X and Y are intervals of Euclidean spaces and Φ is represented by Borel functions [Cesari, Ann. Scuola Norm. Super. Pisa (2) **10**, 253–295 (1941); these Rev. **8**, 257]. *L. Cesari.*

Nöbeling, Georg. Über die Schnittpunkte zweier ebener stetiger Kurven endlicher Länge in allgemeiner Lage. Math. Z. **52**, 637–641 (1950).

Let K_1 and K denote rectifiable plane curves and let K_2 denote the result of translating and rotating K, as a rigid figure, through the vector x, y and the angle ϕ. The author proves that except for a nulset of (x, y, ϕ) the following occurs: (i) the intersection K_1K_2 consists of a finite (possibly empty) set of points p^n; (ii) each p^n corresponds on either curve to a finite set of parameter values only, and for each of these parameter values the curve has a tangent at p^n; (iii) except for possible reversal of directions, the tangents to K_1 at p^n constitute a same line L^{1n} and those to K_2 a same line L^{2n} ($\neq L^{1n}$). *L. C. Young* (Madison, Wis.).

Theory of Functions of Complex Variables

Nagura, Shohei. Faber's polynomials. Kōdai Math. Sem. Rep., no. **5–6**, 5–6 (1949).

If the analytic function $g(z)$ has an expansion of the type $g(z) = z + \sum_{\nu=0}^{\infty} c_\nu z^{-\nu}$ in a neighborhood of $z = \infty$, the Faber polynomials $P_n(z)$ associated with $g(z)$ are characterized by the identities $P_n[g(z)] = z^n + \sum_{\nu=1}^{\infty} c^{\nu(n)} z^{-\nu}$. It was shown by M. Schiffer [Bull. Amer. Math. Soc. **54**, 503–517 (1948); these Rev. **10**, 26] that the Faber polynomials can also be defined by means of the generating function $\log (g(z) - w)/z = -\sum_{\nu=1}^{\infty} \nu^{-1} P_\nu(w) z^{-\nu}$. The author rediscovers this identity and uses it to derive inequalities for functions which are univalent in the unit circle and are subject to certain additional restrictive assumptions. *Z. Nehari.*

Dvoretzky, Aryeh. On sections of power series. Ann. of Math. (2) **51**, 643–696 (1950).

The paper develops the theorem of Jentzsch that every point on the circumference of the circle of convergence of a power series is a cluster point of zeros of its partial sums $s_n(z) = \sum_0^{\infty} a_n z^n$. Only a fraction of the results obtained can be quoted here. The circle of convergence is taken to be $|z| < 1$. It is shown that the image by $w = s_n(z)$ of the neighbourhood of each point of the circumference contains the circle $\log |w| < \delta n$ for infinitely many n. The sequence of values of n, say $n(k)$, can be taken as that of the "large" coefficients ($\lim |a_{n(k)}|^{1/n(k)} = 1$) and the whole phenomenon is then uniform with respect to the different points of the circumference. For the remainders and partial remainders in the neighbourhood of a regular point there are similar results but the value 0 may be exceptional and the circle $\log |w| < \delta n$ needs to be replaced by the annulus $-\delta n < \log w < \delta n$. The w circles are covered $\delta' n$ times where δ and δ' depend on the size of the neighbourhood. Considering variable neighbourhoods in the z-plane it appears that for infinitely many n either $s_n(z)$ or $s_{n-1}(z)$ has zeros in $|z - |a_n|^{-1/n}| < (1+\epsilon) n^{-1} \log n$. Special hypotheses on the order of magnitude of the coefficients (such as $-\infty < \limsup \log |a_n| / \log n < \infty$) and on the behaviour of the function $f(z) = \sum_0^{\infty} a_n z^n$ (such as $\log |f(z)| = o(\log |1-z|)$ for $-\eta_1 < \arg (1-z) < \eta_2$) enable more precise conclusions to be drawn. For example, with the condition on the coefficients, and if $f(z)$ is regular at $z = 1$, then the image by $w = f(z) - s_n(z)$ of $|z-1| < B/n$ covers $\rho < |w| < 1/\rho$ for some positive B. Extensions to Dirichlet series are indicated. There is a critical discussion of the results obtained and their relation to earlier work. The argument is based mainly on the theory of normal families of functions but an alternative for some of the results is the recent work on polynomials by Erdös and Turan [Ann. of Math. (2) **51**, 105–119 (1950); these Rev. **11**, 431]. *A. J. Macintyre* (Aberdeen).

Tsuji, Masatsugu. A remark on Schottky's theorem. J. Math. Soc. Japan **1**, 266–269 (1949).

Let $f(z) = a_0 + a_1 z + \cdots$ be regular, $f(z) \neq 0$ or 1 in $|z| < 1$. The author proves that

$$|f(z)| \leqq \exp \{A \log |a_0| \cdot (1-r) + B \log (|a_0| + 2)/(1-r)\}, \quad |z| = r < 1,$$

and notes that this bound tends to zero with a_0 for fixed r. The proof is based on the classical bound of Bohr and Landau [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1910, 303–330] with $A=0$ in the above and Hadamard's three-circles theorem. The reviewer remarks that, on using his inequality $|a_1|\leqq 2|a_0|(|\log|a_0||+A)$ [Proc. Cambridge Philos. Soc. 43, 442–454 (1947); these Rev. 9, 84] and hence $|f'(z)|\leqq\{2|f(z)|/(1-|z|^2)\}\{|\log(f(z))|+A\}$, and integrating, this could be sharpened to

$$|f(z)|\leqq\exp\left\{\frac{1-r}{1+r}\log|a_0|+A+\frac{1+r}{1-r}\log^+|a_0|\right\}.$$

W. K. Hayman (Exeter).

Okada, Yoshitomo. Some theorems on meromorphic functions. Proc. Amer. Math. Soc. 1, 246–249 (1950).

$F(z)$ étant une fonction méromorphe d'ordre fini ρ, $M(r, F)$ le maximum de $|F(z)|$ pour $|z|=r$ et $N(r)$ le nombre total des zéros et des pôles de $F(z)$ pour $|z|\leqq r$, l'auteur montre que, si ρ n'est pas un entier, on a

$$\liminf_{r\to\infty}\frac{1}{rN(r)}\int_0^r\log^+M(t)dt<\infty;$$

il donne aussi un résultat valable pour ρ quelconque. D'après ses énoncés, l'auteur se bornerait au cas où $F(z)$ est le quotient de deux produits canoniques, mais l'expression de $f(z)$ [bas de la page 247] montre qu'il traite aussi le cas général lorsque ρ n'est pas entier. Par inadvertance, l'auteur définit $M(r, F)$ par le maximum de $|F(z)|$ pour $|z|\leqq r$, maximum qui serait infini en général. L'auteur rapproche ses résultats de ceux obtenus antérieurement pour les fonctions entières par Pólya [Math. Ann. 88, 169–183 (1923)] et Shah [J. London Math. Soc. 15, 23–31 (1940); ces Rev. 1, 307, 400]. Le résultat cité de Pólya est contenu dans ceux du référent [voir Fonctions entières et fonctions méromorphes d'une variable, Mémor. Sci. Math., fasc. 2, Gauthier-Villars, 1925, pp. 26–27] et de Boutroux. *G. Valiron* (Paris).

Morse, Marston. Topological methods in the theory of functions of a complex variable. Ann. Mat. Pura Appl. (4) 28, 21–24 (1949).

This paper is devoted to the statement of a theorem concerning the fundamental relations between the numbers of zeros, poles and branch point antecedents of a function f, where the hypotheses are strictly topological. Let R be the interior of an arbitrary closed Jordan curve of the complex z-plane, let $\bar{R}=R\cup C$, let $w=f(z)$ define a transformation of $\bar{R}$ into the w-sphere subject to the conditions that the transformation is interior over R with at most a finite number of poles, f is continuous over $\bar{R}$ except for its poles in R, and the image g of C is locally simple. Let c be a point of the w-plane not in the image of C, let $n(c)$ be the number of zeros of $f(z)-c$ on R, let $n(\infty)$ be the number of poles of $f(z)$ on R and let μ be the number of branch point antecedents of f on $\bar{R}$. Then if zeros, poles and branch point antecedents are counted with their orders (these orders, as well as the angular order $p(g)$ and classical order $q_c(g)$, are defined topologically) the following fundamental relations are valid: $2n(c)-\mu=1+2q_c(g)-p(g)$, $2n(\infty)=\mu=1-p(g)$.

G. A. Hedlund (New Haven, Conn.).

Behnke, H., und Stein, K. Konvergente Folgen nichtschlichter Regularitätsbereiche. Ann. Mat. Pura Appl. (4) 28, 317–326 (1949).

The authors describe a class of multidimensional Riemann surfaces spread over a complex-projective space which they call "almost finitely-sheeted." The main property proven is that if such a surface is a limit "by exhaustion" of domains of regularity then the limiting surface will also be one.

S. Bochner (Princeton, N. J.).

Rothstein, Wolfgang. Über die Fortsetzbarkeit regulärer und meromorpher Funktionen von zwei Veränderlichen und den Hauptsatz von Hartogs. Math. Nachr. 3, 95–101 (1949).

Étude des domaines de Hartogs qui sont domaines de méromorphie d'une fonction analytique $f(w, z)$, ou qui sont domaines de convergence uniforme d'une suite $f_n(w, z)$ (il n'est pas tenu compte de leur identité avec les domaines d'holomorphie). Citons: si le domaine de convergence uniforme D de $f_n(w, z)$ contient $[|w|<1, |z|<1]$ et si $f_n(c, z)$ converge uniformément pour $w\epsilon E\epsilon[|w|<1]$, $|z|<a$, $a>1$, E étant de mesure harmonique positive dans $|w|<1$, alors D contient $|z|<\Re(w)$, avec $\Re(w)\geqq a$ sur un ensemble de mesure harmonique positive. Pour la méthode suivie, l'auteur se réfère à Lelong [Ann. Sci. École Norm. Sup. (3) 58, 83–177 (1941); ces Rev. 7, 151]; cependant ses résultats peuvent être notablement simplifiés et étendus en utilisant comme au chapitre II du travail cité la notion d'ensemble effilé et le critère de Wiener; le theoreme A de l'auteur est une consequence de l'effilement en a de l'ensemble $w(z)>w(a)+\epsilon$, $\epsilon>0$, $w(z)$ surharmonique; les theoremes 2 et 3 resultent de ce qu'un ensemble (P) n'est effilé qu'en un sous-ensemble de capacité nulle, etc. Une étude des ensembles effilés a été faite par Brelot [J. Math. Pures Appl. (9) 19, 319–337 (1940); Bull. Sci. Math. (2) 68, 12–36 (1944); ces Rev. 3, 47; 7, 15]. *P. Lelong* (Paris).

Hervé, Michel. Sur l'itération des transformations analytiques portant sur deux variables complexes. C. R. Acad. Sci. Paris 230, 1491–1493 (1950).

Verfasser betrachtet die inneren Transformationen $F(z)$ eines beschränkten offenen Gebietes D, ihre Iterierten $F_n(z)$ und deren Grenzabbildungen $\Phi(z)$. Eine analytische Fläche E in D heisst invariant gegenüber $F(z)$, wenn $F(E)\subset E$. Es wird nun $F(z)$ zuerst auf E und dann in D betrachtet und so werden Schlüsse auf die Existenz von Fixpunkten gezogen. Später wird nacheinander von den Voraussetzungen ausgegangen: (1) $F(z)$ hat einen nicht anziehenden Fixpunkt. (2) Wenigstens ein $F_n(z)$ hat einen nicht anziehenden Fixpunkt. (3) $\Phi(z)$ ist eine innere Transformation. Es werden Aussagen über E gemacht. *H. Behnke* (Münster).

Cherubino, Salvatore. Estensione di un lemma di Goursat e funzioni olomorfe di più matrici. Ann. Mat. Pura Appl. (4) 29, 293–299 (1949).

Some remarks on elementary properties of complex analytic functions $f(z_1, \cdots, z_k)$ in which the symbols $z_1, \cdots, z_k$ are not individual complex variables but square matrices of complex variables. *S. Bochner* (Princeton, N. J.).

Special Functions

Richard, Ubaldo. Osservazioni sulla bisezione delle funzioni ellittiche di Weierstrass. Boll. Un. Mat. Ital. (3) 4, 395–397 (1949).

An elementary and direct demonstration of the classical bisection-formula

$$(1)\quad p(u/2)=p(u)+(p(u)-e_1)^{\frac12}(p(u)-e_2)^{\frac12}+(p(u)-e_2)^{\frac12}(p(u)-e_3)^{\frac12}+(p(u)-e_3)^{\frac12}(p(u)-e_1)^{\frac12}$$

from the bisection-equation

$$(2)\quad x^4-4\wp(u)\cdot x^3+\tfrac{1}{2}g_2x^2+(2g_3+g_2\wp(u))x+\tfrac{1}{16}g_2^2+g_3\wp(u)=0.$$

This equation follows from the duplication-formula

$$\wp(2z)=\tfrac{1}{4}\{\wp''(z)/\wp'(z)\}^2-2\wp(z)$$

by elimination of $\wp'(z)$ and $\wp''(z)$ $(u=2z)$. The roots of (1) are $x_0=\wp(\frac{1}{2}u)$, $x_k=\wp(\frac{1}{2}u+\omega_k)$, $k=1, 2, 3$. By the substitution $x=\xi+e_1$, (2) is transformed to a reciprocal equation. By derivation of (1) a bisection-formula for $\wp'(u)$ is obtained.

S. C. van Veen (Delft).

Jackson, M. On well-poised bilateral hypergeometric series of the type ${}_8\Psi_8$. Quart. J. Math., Oxford Ser. (2) **1**, 63–68 (1950).

Using a very general formula of Sears, the author derives two formulae which express an ${}_8\Psi_8$ in terms of Saalschützian ${}_4\Phi_3$'s. One of these, in turn, is used to obtain two-term and three-term relations between well-poised bilateral basic hypergeometric series of type ${}_8\Psi_8$. *N. J. Fine.*

Hahn, Wolfgang. Beiträge zur Theorie der Heineschen Reihen. Die 24 Integrale der hypergeometrischen q-Differenzengleichung. Das q-Analogon der Laplace-Transformation. Math. Nachr. **2**, 340–379 (1949).

The basic hypergeometric series

$$1+\frac{(1-q^\alpha)(1-q^\beta)}{(1-q^\gamma)(1-q)}x+\frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})}{(1-q^\gamma)(1-q^{\gamma+1})(1-q)(1-q^2)}x^2+\cdots$$

satisfies the second-order q-difference equation

$$(1)\quad (q^\gamma-q^{\alpha+\beta+1}x)f(q^2x)-(q^\gamma+q-(q^\alpha+q^\beta)qx)f(qx)\\ -q(1-x)f(x)=0.$$

After introducing some ideas and notations previously used [same vol., 4–34, 263–278 (1949); these Rev. **11**, 29, 356], the author treats the homogeneous first-order q-difference equation. He then develops three types of solutions of (1): Laurent-series, expansions in series of "Potentiellen" $(1+cx)_\nu=\prod_{j=0}^\infty(1+cxq^j)/(1+cxq^{\nu+j})$, ν real, and expansions in terms of $(1+c/x)_\nu$. The last two types are "improper" solutions, that is, they converge only when the series terminate on the right, or when x takes on one of a special sequence of values. Through appropriate choice of the solution-parameters he determines twenty-four "principal solutions" (terminating to the right or left of the zero term, which has coefficient 1), sixteen proper and eight improper; these converge simultaneously in certain domains, and reduce, when $q\to1$, to the 24 Kummer integrals of the hypergeometric differential equation.

The second part of the paper is devoted to relations among the q-analogues of the exponential, circular, Bessel and Whittaker functions; the q-Laplace transform is introduced as a tool, with some indications of a convolution theory. Numerous examples, remarks, and references are given throughout the paper. There is a list of corrections to the first cited paper. *N. J. Fine* (Philadelphia, Pa.).

Pollaczek, Félix. Sur une famille de polynômes orthogonaux qui contient les polynômes d'Hermite et de Laguerre comme cas limites. C. R. Acad. Sci. Paris **230**, 1563–1565 (1950).

By a process of confluence the author derives from certain polynomials introduced by him [same C. R. **228**, 1998–2000 (1949); these Rev. **11**, 104] and the reviewer certain new polynomials $P_n(s;\lambda,\varphi)$ where $\lambda>0$, $0<\varphi<\pi$. These polynomials satisfy the recursion

$$nP_n-2[(n-1+\lambda)\cos\varphi+s\sin\varphi]P_{n-1}+(n-2+2\lambda)P_{n-2}=0,$$

$n=1, 2, \cdots$, $P_0=1$, $P_{-1}=0$. They are orthogonal in $-\infty<s<\infty$ with the weight function $e^{-(\pi-2\varphi)s}|\Gamma(\lambda+is)|^2$. The Laguerre polynomials appear as appropriate limiting cases of P_n. *G. Szegö* (Stanford University, Calif.).

Phillips, R. S., and Malin, Henry. Bessel function approximations. Amer. J. Math. **72**, 407–418 (1950).

In this paper the following bounds for the logarithmic derivates of the modified Bessel and Hankel functions are obtained. For all $v>0$ and $n\geqq1$,

$$\varphi_n\{v,[n(n+1)]^{\frac12}\}<v^{-1}I_n'(v)/I_n(v)<\varphi_n(v,n),$$
$$\varphi_n(v,n)<-v^{-1}K_n'(v)/K_n(v)<\varphi_n\{v,[n(n-1)]^{\frac12}\},$$
$$\varphi_n'(v,n)<[v^{-1}I_n'(v)/I_n(v)]'<\varphi_n'\{v,[n(n+1)^2(n+2)]^{\frac14}\},$$
$$\varphi_n'\{v,[n(n-1)^2(n-2)]^{\frac14}\}<[-v^{-1}K_n'(v)/K_n(v)]'<\varphi_n'(v,n),$$

where $\varphi_n(v,\alpha)=(n/v^2)[1+(v/\alpha)^2]^{\frac12}$. These inequalities result from a study of the behaviour of the functions

$$X_n(v)=v^{-1}I_n'(v)/I_n(v)-\varphi_n(v,\alpha),$$
$$Y_n(v)=-v^{-1}K_n'(v)/K_n(v)-\varphi_n(v,\alpha),$$

which satisfy the Riccati differential equation

$$dZ/dv=\mp vZ^2\mp(2/v)\{n[1+(v/\alpha)^2]^{\frac12}\pm1\}Z\\ \pm v^{-1}\mp(n/\alpha)^2v^{-1}-(n/\alpha^2)v^{-1}[1+(v/\alpha)^2]^{-\frac12}.$$

Combination of the previous results gives

$$I_n'(v)/I_n(v)+K_n'(v)/K_n(v)<0\quad\text{or}\quad[I_n(v)K_n(v)]'<0$$

for all $v\geqq0$ and $n\geqq0$. The case $n=0$ requires a special and somewhat tedious argument. *S. C. van Veen* (Delft).

Differential Equations

Wakerling, Virginia W. The relations between solutions of the differential equation of the second order with four regular singular points. Duke Math. J. **16**, 591–599 (1949).

The paper deals with the ordinary linear differential equation of the second order having four regular singular points. The matter at issue is the determination of the linear relations between the familiar power series solutions relative to any two of the singular points. The author remarks that this problem "has not been seriously studied" nor a method for it given. This ignores the work of W. B. Ford [The Asymptotic Developments of Functions Defined by Maclaurin Series, University of Michigan Press, Ann Arbor, Mich., 1936]. The method, namely of referring the problem to one in difference equations, was also given by Ford.

R. E. Langer (Madison, Wis.).

Levitan, B. M. On a decomposition theorem for characteristic functions of differential equations of the second order. Doklady Akad. Nauk SSSR (N.S.) **71**, 605–608 (1950). (Russian)

The author studies the rate of growth of $\rho(\lambda)$, the Stieltjes distribution function, normalised by $\rho(-0)=0$, occurring in the Parseval formula associated with $y''+(\lambda-q(x))y=0$ over $(0,\infty)$ and with boundary-angle α [cf., e.g., E. C. Titchmarsh [Eigenfunction Expansions . . . , Oxford University Press, 1946, theorem 3.7 and (2.1.4); these Rev. **8**, 458] who, however, writes $k(\lambda)$ for $\pi\rho(\lambda)$]. The results are, for $\mu\to\infty$, (i) if $\cos\alpha=0$, $\rho(\mu)\leqq\frac{1}{2}\pi\mu^{\frac12}+O(\mu^{-\frac12})$,

(ii) if $\sin\alpha=0$, $\rho(\mu)\leqq\frac{1}{2}\pi\mu^{\frac{1}{2}}+O(\mu^{\frac{1}{2}})$, (iii) if $\sin\alpha\cos\alpha\neq 0$, $\rho(\mu)\leqq\frac{1}{2}\pi\csc^2\alpha\mu^{\frac{1}{2}}+O(1)$. The special case $q(x)=0$ shows that these results are precise as regards the power of μ. Finally, tests are derived for the absolute convergence of $\int_{-0}^{\infty}F(\lambda)\phi(x,\lambda)d\rho(\lambda)$. *F. V. Atkinson* (Ibadan).

Roudneff, G. V. Sur les équations de Sturm-Liouville ayant des singularités. Učenye Zapiski Moskov. Gos. Univ. **100**, Matematika, Tom I, 113–126 (1946). (Russian. French summary)

The author considers the Sturm-Liouville problem of $(Ry')'-Qy+\lambda\rho y=0$ where $R(x)$ may vanish at either end of the fundamental interval (a,b), but not in between. If $R(a)=0$, then we must have $R(x)\geqq\alpha(x-a)$ for some $\alpha>0$ and $\int_a^c d\xi/R(\xi)=\infty$, and similarly if $R(b)=0$. Also, $\rho(x)$ may have a finite number of zeros without change of sign in (a,b). The author extends to this case the construction of the resolvent in terms of Neumann series, the variational properties due to Courant [Courant and Hilbert, Methoden der mathematischen Physik, vol. 1, Springer, Berlin, 1924, pp. 345–352], and the result that $\lambda_n=O(n^2)$. Also considered are the Bessel case, and, by a different method, the general case in which $\int_a^c d\xi/R(\xi)$ converges.

F. V. Atkinson (Ibadan).

Pini, B. Autovalori e autosoluzioni per i sistemi autoaggiunti di equazioni differenziali lineari omogenee del secondo ordine. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 351–377 (1949).

This paper is concerned with a self-adjoint two-point boundary problem involving the matrix differential equation $L[y]\equiv(A(x)y'+C(x)y)'+C(x)y'-(P(x)-\lambda Q(x))y=0$, and the end-conditions $y(a)=0=y(b)$, where $A(x)$, $C(x)$, $P(x)$, $Q(x)$ are $n\times n$ matrices of real-valued functions such that on $a\leqq x\leqq b$ the matrices A, P, Q are symmetric and C is skew-symmetric, while the $2n\times 2n$ matrix $\left\|\begin{matrix}A & C\\ -C & P\end{matrix}\right\|$ is positive definite. The author was evidently unaware of a rather extensive literature of the past two decades on problems including the above. For example, his main theorem on the existence of proper values and the oscillation properties of solutions follows from the results of M. Morse [The Calculus of Variations in the Large, Amer. Math. Soc. Colloquium Publ., v. 18, New York, 1934, chapter IV] if one considers $(\pm 1/\lambda)L[y]$, a device suggested initially by Bôcher for the treatment of a single second-order differential equation under comparable hypotheses; moreover, his main theorem is contained as a special case of results of the reviewer on integro-differential systems [Amer. J. Math. **60**, 257–292 (1938)]. Finally, it is to be remarked that the author's discussion of the behavior of points conjugate to $x=a$ as functions of λ appears incomplete in the case of a conjugate point of multiplicity greater than one.

W. T. Reid (Evanston, Ill.).

Stiefel, Eduard, und Ziegler, Hans. Natürliche Eigenwertprobleme. I. Z. Angew. Math. Physik **1**, 111–138 (1950).

The authors set themselves the task of developing a general treatment of eigenvalue problems including those in which the eigenvalues appear explicitly in the boundary conditions. They feel that from the point of view of the applications to stability and vibrational questions in mechanics the variational approach is the most suitable one (as compared with the approach by differential or integral equations). In the present first communication a number of properties are discussed which are satisfied in many of the above applications and whose validity can easily be tested in each concrete case. A variational problem having these properties is called a natural eigenvalue problem. In addition, the differential equation and boundary conditions corresponding to a natural eigenvalue problem are formulated. Only the case of one dependent and one independent variable is treated. The differential equation may be of arbitrary even order. The proof of the classical theorems of the eigenvalue theory for the natural eigenvalue problems and the formulation of the problem in terms of an integro-differential equation is left to further communications. *E. H. Rothe.*

Brillouin, L. The B.W.K. approximation and Hill's equation. II. Quart. Appl. Math. **7**, 363–380 (1950).

Der Verf. geht zunächst auf die Grundlagen der B.W.K.-Approximation im allgemeinen ein und insbesondere auf die Möglichkeit, die erste Näherung dieser Approximation durch weitere Näherungen zu verbessern. Er zeigt in einem Vergleich der Approximationsergebnisse mit der asymptotischen Entwicklung der Hankelschen Funktionen, dass die aus der B.W.K.-Näherung erhaltene Reihenformel im allgemeinen halbkonvergent sein muss. Weiter zeigt sich, dass die Näherung eine gute sein kann ähnlich wie jene der Debyeschen Reihen. Verf. wendet nun seine Formeln auf die Mathieusche Differentialgleichung an und erhält einige Ergebnisse bezüglich der charakteristischen Exponenten, welche mit dem Theorem von Floquet zusammenhängen. Leider nimmt der Verf. keinen Bezug auf die ausführlichen Arbeiten von R. E. Langer über die Anwendungen der B.W.K.-Methode [z.B., Trans. Amer. Math. Soc. **37**, 397–416 (1935)]. Weiterhin hat der Ref. dieses Verfahren in einer Reihe von Arbeiten auf Hillsche Differentialgleichungen angewandt, von denen nur zwei zitiert seien [Math. Z. **49**, 593–643 (1944); Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 278–296 (1948); diese Rev. **6**, 174; **10**, 41]. Diese Arbeiten werden vom Verf. nicht berücksichtigt. *M. J. O. Strutt* (Zürich).

Ghizzetti, Aldo. Un teorema sul comportamento asintotico degli integrali delle equazioni differenziali lineari omogenee. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 28–42 (1949).

Consider the linear differential equation $F(D)y=\phi(D)y$ where $D=d/dx$, $F(D)=a_0+a_1D+\cdots+a_{n-1}D^{n-1}+D^n$, $\phi(D)=\phi_0(x)+\phi_1(x)D+\cdots+\phi_{n-1}(x)D^{n-1}$. Since the early work of Dini and Poincaré, a problem of recurrent interest has been that of determining the precise sense in which the "smallness" of the $\phi_k(x)$ as $x\rightarrow\infty$ assures that the asymptotic behavior of y as $x\rightarrow\infty$ is identical with that of the solution of $F(D)z=0$. Recent work on the problem has been done by Faedo [Ann. Mat. Pura Appl. (4) **25**, 111–133 (1946); **26**, 207–215 (1947); these Rev. **9**, 285; **10**, 120; cf. also the reviewer, Duke Math. J. **14**, 83–97 (1947); these Rev. **9**, 35; where further references will be found]. The author limits himself in this paper to the case $F(D)=D^n$ and shows that if $\int^\infty x^{n-r-1}|\phi_r(x)|dx<\infty$, $r=0,1,\cdots,n-1$, there exist n independent integrals $y_k(x)$ such that

$$\lim_{x\rightarrow\infty} y_k^{(s)}(x)/x^{k-s}=\begin{cases}1/(k-s)!, & s=0,1,\cdots,k,\\ 0, & s\geqq k.\end{cases}$$

R. Bellman (Stanford University, Calif.).

Massera, J. L. On Liapounoff's conditions of stability. Ann. of Math. (2) **50**, 705–721 (1949).

A stable solution $x(t)=(x_1(t),\cdots,x_n(t))$, say $x=0$ for simplicity, of a system $\dot{x}(t)=X(x,t)$ of ordinary differential

equations is called by Liapounoff asymptotically stable if there exists a $\delta>0$ such that every solution whose initial conditions at $t=0$ differ from 0 by less than δ tends to 0 as t tends to infinity. The present author introduces the stronger notion of equi-asymptotic stability by requiring the existence of a positive δ such that the $\lim x(t)=0$ as $t\to\infty$ be uniform for all initial conditions $|x(0)|\leqq\delta$. The relationship between these two notions of asymptotic stability is investigated. In general they are different, but coincide for systems which are either of order 1, linear, periodic, or autonomous. Several known theorems on asymptotic stability are strengthened, both by weakening the assumptions and by showing that equi-asymptotic stability is implied. These results lead to necessary and sufficient conditions for asymptotic stability when the system is either linear or periodic. *F. Bohnenblust* (Pasadena, Calif.).

Cartwright, M. L. Forced oscillations in nonlinear systems. Contributions to the Theory of Nonlinear Oscillations, pp. 149–241. Annals of Mathematics Studies, no. 20. Princeton University Press, Princeton, N. J., 1950. $4.00.

The author discusses informally a number of solved and unsolved problems associated with

$$(*)\qquad \ddot{x}-k(1-x^2)\dot{x}+x=bk\lambda\cos(\lambda t+\alpha)$$

and generalizations of (*). The early sections are an exposition of relevant topological background and of analytic devices used to show that the topological transformations associated with the differential equation satisfy various required hypotheses of a rather general nature. Next, nearly linear oscillations are considered with an account of known methods as well as the more recent work of the author and Littlewood [cf. Cartwright, Proc. Cambridge Philos. Soc. 45, 495–501 (1949); these Rev. 11, 249] using difference equations. A section is devoted to questions concerning rotation numbers. Finally, the aspects of the singular case, $k\to\infty$, are discussed for a generalized form of (*). *N. Levinson* (Cambridge, Mass.).

Levinson, Norman. Perturbations of discontinuous solutions of non-linear systems of differential equations. Acta Math. 82, 71–106 (1950).

Let x be an n-dimensional vector and let u be a scalar. The system of differential equations $\dot{x}=f\dot{u}+\varphi$; $\epsilon\ddot{u}+g\dot{u}+h=0$ is considered for $\epsilon>0$. The functions f, φ are vector functions depending on x, u, t, ϵ and g, h are scalar functions of the same variables. For $\epsilon=0$, the system degenerates to $\dot{y}=f\dot{v}+\varphi$, $g\dot{v}+h=0$, where x, u are replaced by y, v in order to distinguish between the two systems. Since g may vanish at certain points, the solutions of the degenerate system may have discontinuities. Functions $y(t)$, $v(t)$, $\alpha\leqq t\leqq\beta$ are called a solution if (1) with the exception of a finite number of values τ_j of t, $\alpha<\tau_1<\cdots<\tau_m<\beta$, y and v are continuous and the limits of y and v exist at $\tau_j\pm0$, (2) in each of the subintervals, y and v satisfy the degenerate system, (3) at $t=\alpha,\beta$ the function g is greater than 0 and in each subinterval g is greater than 0 and tends to zero as $t\to\tau_j-0$, (4) the jumps at each τ are obtained by solving $dy/dv=f(y, v, \tau, 0)$ from $v(\tau-0)$, $y(\tau-0)$ for increasing or decreasing v, depending on the sign of h, until the first value of v for which the integral $\int g(y(v), v, \tau, 0)dv$, extended from $v(\tau-0)$ to v, vanishes. This value of v is $v(\tau+0)$ and the corresponding value of y is $y(\tau+0)$. It must be assumed that $\sum f_i\partial g/\partial y_i+\partial g/\partial v\neq0$ at $\tau-0$ and that h does not vanish there. It is shown that, given a solution of the degenerate system as described above with the initial conditions $y(\alpha)$, $v(\alpha)$, then for ϵ sufficiently small and any initial conditions sufficiently near to $y(\alpha)$, $v(\alpha)$ the solution of the original system exists in $\alpha\leqq t\leqq\beta$ and, excluding arbitrary neighborhoods of the τ-values, the solution tends uniformly to $y(t)$, $v(t)$ as $\epsilon\to0$ and as the initial conditions tend to $y(\alpha)$, $v(\alpha)$. Additional theorems discuss a similar behavior for the derivatives of x and u with respect to the initial conditions. *F. Bohnenblust* (Pasadena, Calif.).

Levinson, Norman. An ordinary differential equation with an interval of stability, a separation point, and an interval of instability. J. Math. Physics 28, 215–222 (1950).

The special differential equation $\epsilon d^2u/dx^2+f(u, du/dx)=0$ is considered, where $f=u-[4u'/(3+u'^4)]$. For the fixed initial conditions $x=0$, $u=8/19$, $u'=2$, the behavior of the solution $u=u(x,\epsilon)$ is compared with that of the solution $z=z(x)$ of the limiting differential equation $f(z, z')=0$ which satisfies the same initial conditions. The existence of a value $x_1>0$ is shown such that the solution $z(x)$ exists in $0\leqq x\leqq x_1$ but cannot be prolonged beyond this value. On the other hand, the solution $u(x,\epsilon)$ tends to $z(x)$, as ϵ tends to $+0$, in the interval $0\leqq x<x_1$, but for $x>x_1$ it oscillates more and more rapidly, as ϵ tends to $+0$, with an amplitude approximately equal to one. In particular, it cannot tend to a limit as ϵ tends to zero. In the study of viscous incompressible flows around an obstacle, experiments have suggested the occurrence of regions of stability and of instability. Although the example considered in this paper has no direct bearing on the problem of viscous flows, its interest lies in the fact that it exhibits a simple ordinary differential equation in which such intervals of stability and of instability occur. *F. Bohnenblust* (Pasadena, Calif.).

Manacorda, Tristano. Sopra un'equazione differenziale non lineare della dinamica del punto. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 11(80) (1947), 85–98 (1949).

The behavior of the solutions of the nonlinear equation $\ddot{y}+2\alpha\dot{y}+f(y,t)y=0$ is studied by the classical methods of the theory of Sturm. It is assumed that $|f|$ is bounded, which implies the existence of the solutions for all values of t. The two main results are: if there exists an ω such that $f(y,t)\leqq\omega^2<\alpha^2$, then every solution y can vanish at most once; if there exists ω such that $0\leqq\alpha^2<\omega^2\leqq f(y,t)$, then every solution has infinitely many zeros. The author restricts himself unnecessarily to a more special form of $f(y,t)$. It is erroneously stated on page 89 under 4a that the solutions y do not vanish for $t>0$. [In the notation of the author the case $\beta=0$, $\gamma=$ constant, $\gamma<\alpha^2/4$ leads to a counterexample. The differential equation becomes the linear equation $\ddot{y}+\alpha\dot{y}+\gamma y=0$ whose solutions may well vanish for $t>0$.] As stated above it is only true, under the conditions assumed, that they vanish at most once. As a result of this error some of the later proofs must be changed slightly. *F. Bohnenblust* (Pasadena, Calif.).

Nardini, Renato. Sulla stabilità delle vibrazioni quasi armoniche di un sistema dissipativo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 603–608 (1949).

The constant c is positive, the function $w(t)$ is continuously differentiable, periodic with period T, has k relative maxima in a period and is positive. The solution $y=0$ of $\ddot{y}+2c\dot{y}+w^2(t)y=0$ is called stable if every solution and its derivative tend to zero as $t\to\infty$. It is shown that stability

occurs if $2c > \max |\dot{w}|/w$. Stability also occurs if $c < m$ and $2cT > k \log (M^2-c^2)/(m^2-c^2)$, where m, M are the minimum and the maximum of w. *F. Bohnenblust.*

Nardini, Renato. Sulle vibrazioni quasi-armoniche di un sistema dissipativo con elasticità periodica. Boll. Un. Mat. Ital. (3) 4, 370–373 (1949).

Same problem as in the paper reviewed above. The condition $c < m$ is removed and a similar but more complicated inequality is obtained showing that if T is sufficiently large then stability takes place. *F. Bohnenblust.*

Amerio, Luigi. Determinazione delle condizioni di stabilità per gli integrali di un'equazione interessante l'elettrotecnica. Ann. Mat. Pura Appl. (4) 30, 75–90 (1949).

The author studies the nonlinear differential equation $y''+ay'+b \sin y = c$ proposed by E. Bottani in connection with an engineering problem. Particular results have been obtained by Tricomi [Ann. Scuola Norm. Super. Pisa (2) 2, 1–20 (1933)]. The paper begins with a study of the conditions to be satisfied by a, b, c in order that there exist a solution y asymptotic to a constant. The change of variable $y'=p$ converts the equation into a first order equation to which the methods of Poincaré are applied. By this means the question of the existence of a periodic solution is treated. *R. Bellman* (Stanford University, Calif.).

Rosenblatt, Alfredo. On the phenomenon of subresonance for the van der Pol equation. Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadística 1, no. 8 = Bol. Fac. Ingen. Montevideo 3, no. 3, 12 pp. (1949). (Spanish)

Van der Pol's equation $\ddot{x}+x+\mu\dot{x}(A+Cx^2)=e \cos \omega t$, $A<0$, $C>0$, with a forcing term, is investigated for small μ when the frequency of the forcing term is an integral multiple of the natural frequency of the system, i.e., $\omega=n>1$. The substitution of $x=x_0+\mu x_1+\mu^2 x_2^2+\cdots$ in the equation determines the x_k successively, each up to an additive term of the form $a_k \sin t+b_k \cos t$. Can these constants be so determined as to make each x_k a periodic function with period 2π? For $n=3$ the answer is shown to be affirmative if and only if $-256A/C \geqq 7e^2$. For $n\neq 3$, the determinants of the linear equations which determine the a_k, b_k vanish and the construction breaks down. *F. Bohnenblust.*

Gonzalez Baz, Enriqueta. Relation between the parameter and the dimensions of the periodic solution of the van der Pol equation. Comisión Impulsora y Coordinadora de la Investigación Científica (Mexico). Anuario 1947, 87–95 (1949). (Spanish)

The system of differential equations $3\dot{x}=3y-\mu x^3+3\mu x$, $\dot{y}=-x$ is equivalent to van der Pol's equation. It has only one periodic solution which over a range of values of the parameter μ is close to a circle of radius k. The relation between k and μ is computed explicitly by introducing polar coordinates and taking $r=k$ in the resulting expression for the derivative $dr/d\theta$. The relation is obtained from $\int dr=0$ extended over $-\pi/2<\theta<\pi/2$. The exact significance of k is not discussed; it is not clear, for example, whether k lies between the maximal and the minimal distance of the true solution from the origin. *F. Bohnenblust.*

Ludeke, Carl A. A method of equivalent linearization for non-linear oscillatory systems with large non-linearity. J. Appl. Phys. 20, 694–699 (1949).

This paper contains a report based partially on the existing theory of nonlinear equations and partially on experimental work. The periodic solutions of period $2\pi/\omega$ of the system $\ddot{x}+\omega_0^2 x+bx^3=F \cos \omega t$ are investigated in their dependence on the coefficient b which is used as a measure of the nonlinearity. This coefficient is not restricted to small values. The results indicate that only the first two terms of the Fourier series expansion of the solution are important, i.e., the terms in $\cos \omega t$ and $\cos 3\omega t$. The frequency-amplitude relation seems well approximated by an expression of the form $\omega^2=\omega_0^2+cbA^2-F/A$; where A stands for the maximal value of the solution. Although the coefficient c varies with b the value $c=.9$ is a good average. Further experimental results are discussed which concern the same equation but with a damping term. *F. Bohnenblust.*

Popovskiĭ, A. M. On the freedom of choice of the parameters of autonomic processes of regulation of several reciprocally related quantities. Avtomatika i Telemehanika 10, 401–423 (1949). (Russian)

The author sets up the equations governing linear control mechanisms which regulate several variables simultaneously. *E. N. Gilbert* (Murray Hill, N. J.).

Solodovnikov, V. V. The frequency-response method in the theory of regulation (a survey). Avtomatika i Telemehanika 8, 65–88 (1947). (Russian)

Kallmann, H., und Päsler, M. Zur Integration der gestörten zeitabhängigen Schrödinger-Gleichung. Z. Physik 126, 749–759 (1949).

The authors consider the time dependent Schroedinger equation (*) $[H+\lambda H_s+(h/i)\partial/\partial t]\Psi=0$, where the solutions of the problem $(H+(h/i)\partial/\partial t)\Phi=0$ are known to be $\Phi_n=e^{-iE_nt/h}\varphi_n(x,y,z)$ and $H\varphi_n=E_n\varphi_n$. Taking the Laplace transform of (*), one obtains $(H+\lambda H_s+(h/i)p)g=(h/i)\psi_0$, where ψ_0 is the initial value of Ψ. The function g is expanded in terms of φ_n and the coefficients are determined by the perturbation method. The results are applied to obtain $\Psi^*\Psi$ to the same approximation. *H. Feshbach.*

Durand, Émile. Sur la résolution de l'équation radiale des atomes hydrogénoïdes. C. R. Acad. Sci. Paris 230, 273–275 (1950).

The author solves the equation

$$\varphi[\tfrac{1}{4}+l(l+1)/l^2-d^2/d_l^2]S_{n,l}=nS_{n,l}$$

by the method of factorization. *H. Feshbach.*

de Wet, J. S., and Mandl, F. On the asymptotic distribution of eigenvalues. Proc. Roy. Soc. London. Ser. A. 200, 572–580 (1950).

The authors consider the asymptotic distribution of eigenvalues for the Schroedinger equation (*) $\nabla^2\psi+[\lambda-q(\mathbf{r})]\psi=0$. For certain types of potential q they show that if $A(\lambda)$ is the number of eigenvalues of (*) less than λ, then $|A(\lambda)-I(\lambda)|\leqq R(\lambda)$. The function $I(\lambda)$ is

$$I_p(\lambda)=C_p\int_D[\lambda-q(\mathbf{r})]^{p/2}d\mathbf{r},$$

where p denotes the number of dimensions involved, $C_1=1/\pi$, $C_2=1/4\pi$, $C_3=1/6\pi^2$. The domain of integration D is bounded by the surface defined by $\lambda=q(\mathbf{r})$. For a wide class of potentials q, $R(\lambda)/I(\lambda)\to 0$ as $\lambda\to\infty$. *H. Feshbach.*

Fichera, Gaetano. Analisi esistenziale per le soluzioni dei problemi al contorno misti, relativi all'equazione e ai sistemi di equazioni del secondo ordine di tipo ellittico, autoaggiunti. Ann. Scuola Norm. Super. Pisa (3) 1 (1947), 75–100 (1949).

L'autore studia, per l'equazione lineare di tipo ellittico

$$(1)\qquad E[u]=\sum_{i,k=1}^{m} a_{ik}\frac{\partial^2 u}{\partial x_i\partial x_k}+\sum_{k=1}^{m} b_k\frac{\partial u}{\partial x_k}+cu=f\qquad (c\leqq 0)$$

autoaggiunta, il seguente problema misto e ne dà una soluzione. Sia D il dominio di integrazione della (1) e supponiamo assegnate: la funzione $u(Q)=\alpha(Q)$ su una parte, F_1D, della frontiera FD, la derivata conormale $\partial u/\partial\nu=\beta(Q)$ sulla rimanente parte F_2D, di FD. Si può supporre $\alpha\equiv 0$, $f\equiv 0$. Sia: D' un dominio regolare contenente D e tale che risulti $FD\cdot FD'=F_1D$; $G(Q,P)$ la funzione di Green per il problema di Dirichlet relativo alla (1) in D'; $\{\varphi_k(P)\}$ una successione hilbertianamente completa in $D'-D$; $\delta^*(Q)$ una funzione definita su FD e coincidente con $\delta(Q)$ su F_2D (e tale che sia $\int_{FD}\delta^*(Q)d\sigma=0$ se $c\equiv 0$); u^* la soluzione del problema di Neumann con la condizione $\partial u^*/\partial\nu=\delta^*$. Posto allora $v^{(k)}(P)=\int_{D'-D}\varphi_k(P)G(P,P')d\tau$, sarà $v^{(k)}(P)=0$ su F_1D. Consideriamo per l'equazione $E[u]=0$ la formula di Green:

$$-\int_D\{\Gamma[u]\times\Gamma[u']\}d\tau=\int_{FD}u(\partial u'/\partial\nu)d\sigma=\int_{FD}u'(\partial u/\partial\nu)d\sigma$$

dove $\Gamma[u]$ è un noto vettore a $m+1$ componenti, combinazioni lineari di u e della sue derivate parziali prime. Possiamo supporre la successione $\Gamma[v^{(k)}]$ ortonormale in D. Posto allora

$$c_k=\int_{FD}v^{(k)}\delta^*d\sigma=\int_{F_2D}v^{(k)}\delta d\sigma,$$

si ha anche

$$c_k=-\int_D\{\Gamma[u^*]\times\Gamma[v^{(k)}]d\tau$$

e quindi la serie $\sum_0^\infty c_k{}^2$ converge. Perciò la successione

$$-\sum_0^n c_k\Gamma[v^{(k)}]=\Gamma\left(-\sum_0^n c_kv^{(k)}\right)$$

converge in media, di qui si deduce la convergenza della serie $\sum c_kv^{(k)}$ e la soluzione del problema. Nel questo lavoro l'autore estende il procedimento indicato anche allo studio dei sistemi, in particolare al sistema delle equazioni dell'elasticità. *L. Amerio* (Milan).

Lavrent'ev, M. A., and Bicadze, A. V. On the problem of equations of mixed type. Doklady Akad. Nauk SSSR (N.S.) 70, 373–376 (1950). (Russian)

For each number k such that $0<k\leqq 1$, let D_k be the open set in the (x,y)-plane whose boundary is $L+L_k+$part of L_2, where L is a smooth Jordan arc, situated in the upper half plane $y\geqq 0$ save for its end points $A=(0,0)$ and $B=(1,0)$, L_k is the straight line segment, $0\leqq x\leqq(1+k)^{-1}$, $y=-kx$, and L_2 is the straight line segment, $\frac{1}{2}\leqq x\leqq 1$, $y=x-1$. The following mixed boundary value problem is considered: to determine a solution $u(x,y)$ of the equation $u_{xx}+\theta(y)u_{yy}=0$ (where the step function $\theta(y)$ equals 1 for $y>0$ and -1 for $y<0$) in the open set D_k minus the open segment AB of the x axis joining A and B, the function u having continuous first partial derivatives in the closure of D_k, save perhaps at A and at B, and satisfying the boundary conditions $u=\varphi$ on L and $u=\psi$ on L_k, where φ and ψ are given functions such that $\varphi(A)=\psi(A)$. For $k=1$ this boundary value problem is akin to that dealt with by Tricomi [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (5) 14, 134–247 (1924)] for the equation $yu_{xx}+u_{yy}=0$. (The authors also discuss a more complicated boundary value problem for the case $k=1$, which they call the "generalized Tricomi problem.") Let D be the open subset of D_k which is bounded by L and by the straight line segment AB of the x axis. Under suitable conditions, the problem for $k=1$ can be reduced to the determination of a function harmonic in D, having prescribed boundary values on L and a prescribed directional derivative (in a fixed direction) on AB, and by a subsequent conformal mapping to an ordinary Dirichlet problem. For k not necessarily 1, the boundary value problem is reduced to the determination of a function harmonic in D, having prescribed boundary values on L, and whose tangential and normal derivatives satisfy a certain relation on AB. Finally, analytic solutions, corresponding to analytic boundary values φ and ψ, are discussed. *J. B. Dias* (College Park, Md.).

Bicadze, A. V. On some problems of mixed type. Doklady Akad. Nauk SSSR (N.S.) 70, 561–564 (1950). (Russian)

In the paper reviewed above certain boundary value problems of mixed type were shown to possess one and only one solution. The present paper deals with the explicit determination of the solution for special types of domains. In particular, the "generalized Tricomi problem" of the above paper is dealt with in the case when the Jordan arc L [see the preceding review for notation] is taken to be the semicircle $x^2+y^2-x=0$, $y\geqq 0$. The actual determination of the solution in this case leads to a Riemann-Hilbert problem [see N. I. Mushelišvili, Singular Integral Equations . . . , OGIZ, Moscow-Leningrad, 1946; these Rev. 8, 586]. The solution of another related boundary value problem, different from the boundary value problems of the paper reviewed above, is also given explicitly for the same special domain. *J. B. Dias* (College Park, Md.).

Germay, R. H. J. Sur la fonction de Riemann associée à l'équation aux dérivées partielles du second ordre, de forme linéaire, du type hyperbolique. Bull. Soc. Roy. Sci. Liége 18, 383–390 (1949).

The Riemann function for the limit of a sequence of equations is found from an iteration scheme directly involving the coefficients of the equations of the sequence, instead of their limits. *F. John* (Los Angeles, Calif.).

Germay, R. H. Sur des équations récurrentes aux dérivées partielles du second ordre. Bull. Soc. Roy. Sci. Liége 18, 474–480 (1949).

Under conditions too detailed to state here in their entirety, the author proves by use of the method of successive approximations the existence of a unique solution (z_n) of the recurrent partial differential system

$$\frac{\partial^2 z_n}{\partial x\partial y}=F_n\left(x,y,z_n,\frac{\partial z_n}{\partial x},\frac{\partial z_n}{\partial y},\cdots,z_{n+m-1},\frac{\partial z_{n+m-1}}{\partial x},\frac{\partial z_{n+m-1}}{\partial y}\right),$$

which reduces to $(a_n(x))$ for $y=0$ and $(b_n(y))$ for $x=0$. The result is an extension of the work of Bruwier concerning the recurrent differential system $y_n'=F_n(x,y_n,y_{n+1})$ [Mém. Soc. Roy. Sci. Liége (3) 17, no. 22 (1932)]. *R. Bellman* (Stanford University, Calif.).

Castoldi, Luigi. Un teorema di media per le soluzioni regolari dell'equazione $\Delta\Delta u=\lambda^2u$. Atti Accad. Ligure 5, 135–142 (1949).

L'autore, utilissando precedenti risultati di Pizzetti [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 18_1, 182–

185 (1909)] e Sbrana [ibid. (6) 1, 369–371 (1925)] dimostra che, se u è soluzione dell'equazione $\Delta^2 u=\lambda^2 u$, la media $M(P,r)$ dei valori assunti da u su una superficie sferica di centro P e raggio r è caratterizzata dalle relazione

$$M(P,r)=u(P)\,U_n(\tfrac{1}{2}r\lambda^{\frac{1}{2}})+\lambda^{-1}\Delta u(P)\cdot Z_n(\tfrac{1}{2}r\lambda^{\frac{1}{2}}),$$

con U_n e Z_n trascendenti intere collegate alle funzioni di Bessel. *L. Amerio* (Milano).

Lopatinskiĭ, Ya. B. A fundamental system of solutions of a system of linear differential equations of elliptic type. Doklady Akad. Nauk SSSR (N.S.) 71, 433–436 (1950). (Russian)

Consider the system of linear partial differential equations (*) $\sum_{l=1}^{p} A_{kl}u_l=0$, $k=1,\cdots,p$, where

$$A_{kl}=\sum_{0\leqq j_1+\cdots+j_n\leqq s_{kl}} f^{kl}_{j_1,\cdots,j_n}\frac{\partial^{j_1+\cdots+j_n}}{\partial x_1^{j_1}\cdots\partial x_n^{j_n}},$$

s_{kl} is the order of A_{kl}, and each coefficient $f^{kl}_{j_1,\cdots,j_n}$ is a real-valued function defined in an open set D of real $(x_1,\cdots,x_n)$-space, $n\geqq 2$. The system (*) is supposed to be of elliptic type, i.e., for each $(x_1,\cdots,x_n)$ in D and each sequence of real numbers $\alpha_1,\cdots,\alpha_n$ such that $\alpha_1^2+\cdots+\alpha_n^2>0$, the determinant

$$\left|\sum_{j_1+\cdots+j_n=s_l} f^{kl}_{j_1,\cdots,j_n}(x_1,\cdots,x_n)\alpha_1^{j_1}\cdots\alpha_n^{j_n}\right|\neq 0,$$

where $s_l=\max_k s_{kl}$ for $l=1,\cdots,p$. Under these hypotheses, the author announces the following theorem. If the coefficients of the system (*) are continuously differentiable $\sum_{l=1}^{p}s_l$ times, then the system (*) possesses a system of fundamental solutions in D. For the case $n=2$, using additional hypotheses, this theorem was proved by E. E. Levi [Rend. Circ. Mat. Palermo 24, 275–317 (1907)]. It is stated that in the proof of the theorem use is made of an explicit formula given by the author for a fundamental solution of the equation $f(\partial/\partial x_1,\cdots,\partial/\partial x_n)u=0$, where $f(\alpha_1,\cdots,\alpha_n)$ is a definite form of positive degree in $\alpha_1,\cdots,\alpha_n$ with constant coefficients. In the special case of operators A_{kl} of first order, and $n=3$, a fundamental system of solutions has been obtained in another manner by Shapiro [C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 133–135 (1945); these Rev. 7, 14]. *J. B. Diaz* (College Park, Md.).

Cinquini-Cibrario, Maria. Sui sistemi di equazioni alle derivate parziali di ordine superiore. Ann. Mat. Pura Appl. (4) 29, 147–161 (1949).

The author considers a system of m equations in m unknown functions $z_1(x,y),\cdots,z_m(x,y)$ of the form

$$(1)\qquad F_i(x,y;z_j;p_{j,r_j,s_j})=0,\qquad i=1,\cdots,m.$$

Here $p_{j,r_j,s_j}=\partial^{r_j+s_j}z_j/\partial x^{r_j}\partial y^{s_j}$, $r_j+s_j=1,\cdots,n_j$. For given z, p a curve C with equation $y=f(x)$ is characteristic, if the equations obtained from (1) by differentiation are insufficient to express all derivatives of the z_j of order n_j+1 in terms of derivatives along C. The curve C is characteristic if dy/dx coincides with a root ρ of an algebraic equation of the form $\Delta=0$, where Δ is a certain determinant of order $n_1+\cdots+n_m$. The system (1) is called hyperbolic, if all roots ρ are real and distinct for x,y,z_j,p_{j,r_j,s_j} in a domain D. The Cauchy problem for (1) consists in finding a solution of (1), for which the z_j and p_{j,r_j,s_j} take prescribed values along a curve C, which are compatible with each other and with (1). It is proved that under certain regularity conditions the Cauchy problem has a unique solution in the small, provided C is not characteristic with respect to the data. The method of proof consists in replacing (1) by an equivalent system of first order equations for which the Cauchy problem has been solved previously by the author [Rend. Sem. Mat. Univ. Padova 17, 75–96 (1948); Atti Acad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 4, 682–688 (1948); these Rev. 10, 539, 608]. It is shown that the assumption that (1) is hyperbolic can be replaced by the weaker condition that the roots of $\Delta=0$ are all real and of constant multiplicity in D, and that for a root of multiplicity h the rank of Δ is $m-h$. *F. John.*

Difference Equations, Special Functional Equations

Schmidt, Hermann. Über Wurzelapproximation nach Euler und Fixgebilde linearer Transformationen. Math. Z. 52, 547–556 (1950).

The author discusses the system of homogeneous linear first order difference equations with constant coefficients $x(t+1)=x(t)A$, where x is an m-vector and A is a non-singular $m\times m$ matrix of complex numbers. A solution which satisfies the initial condition $x(0)=x_0$, and is analytic for all real t, is found in the form $x(t)=x_0A^t$, where $A^t=(2\pi i)^{-1}\int_C z^t(zE-A)^{-1}dz$ for a suitable C. By use of the calculus of residues an asymptotic formula for the solution is found at once. The methods are applied to two special cases of A, one corresponding to Euler's method for extracting roots [see Müller, Math. Z. 51, 474–496 (1948); these Rev. 10, 574], the other yielding a generalization of Bernouilli's method for approximating roots of algebraic equations. When the characteristic values of A satisfy certain conditions the asymptotic formula is interpreted in terms of the "attractiveness" of a fixed point of a linear transformation in $(m-1)$-dimensional projective space. Finally, it is shown that every open Jordan curve in m-dimensional affine space, which is fixed under the transformation $x\to xA$, is given by $x=p(t)A^t$, where $p(t)$ is a suitable continuous periodic vector of period 1. *W. Strodt.*

Hartman, Philip, and Wintner, Aurel. On linear difference equations of second order. Amer. J. Math. 72, 124–128 (1950).

Les auteurs donnent deux théorèmes sur les équations aux différences du deuxième ordre. (I) Les hypothèses $1-r_k-q_k>0$, $q_k>0$ $(k=0,1,2,\cdots)$ entraînent que l'équation $\Delta^2y_k+r_k\Delta y_k-q_ky_k=0$ ait une solution y_k telle que $y_k>0$, $\Delta y_k<0$. (II) Les hypothèses $p_k>0$, $p_k-r_k-q_k>0$, $q_k>0$, $(-1)^n\Delta^{n+1}p_k\geqq 0$, $(-1)^n\Delta^nq_k\geqq 0$, $(-1)^n\Delta^nr_k\geqq 0$ $(k,n=0,1,2,\cdots)$ entraînent que l'équation $p_k\Delta^2y_k+r_k\Delta y_k-q_ky_k=0$ ait une solution y_k telle que $y_k>0$, $(-1)^n\Delta^ny_k\geqq 0$. *A. Ghizzetti* (Rome).

Stephens, C. F. Nonlinear difference equations containing a parameter. Proc. Amer. Math. Soc. 1, 276–281 (1950).

Consider the system of nonlinear difference equations

$$(1)\qquad y_i(x+1)=x^{-1}\sum_{j=1}^{n}b_{ij}(x)y_j(x)+f_i[y_1(x),\cdots,y_n(x);p;x],$$

$f_i(0,\cdots,0;0;x)\equiv 0$, $i=1,2,\cdots,n$, where the f_i are analytic functions of y_i and the constant parameter p, continuous functions of the complex variable x, and bounded for all values of the $y_i(x)$, p, and x for the domain G: $|y_i(x)|\leqq r_i$, $|p|\leqq\rho$, $|x|\geqq K$, and the $b_{ij}(x)$ are any continuous and bounded functions of x which are defined in G. The system is shown to have a solution which is analytic in p and continuous in x. *D. Moskovitz.*

Myškis, A. Investigation of a class of differential equations with retarded arguments by means of a generalized Fibonacci series. Doklady Akad. Nauk SSSR (N.S.) 71, 13–16 (1950). (Russian)

The author studies the equation $y'(x)+M(x)y(x-\Delta(x))=0$ [$M(x)\geqq 0$, $\Delta(x)\geqq 0$; $A\leqq x<B$]. Use is made of the definitions and notation in an earlier paper [same Doklady (N.S.) 70, 953–956 (1950); these Rev. 11, 522]. It is assumed that $M_0<\infty$, $\Delta_0<\infty$, $0<M_0\Delta_0<\frac{1}{4}$. Existence and uniqueness of a solution $y(x)$, subject to assigned initial conditions $\varphi(x)$, have been proved by the author previously [Uspehi Matem. Nauk (N.S.) 4, no. 5(33), 99–141 (1949); these Rev. 11, 365]. If there are no zeros of $y(x)$ for $x\geqq A$, let $b=B$; in the contrary case let b be the least zero of $y(x)$ for $x\geqq A$. Suppose $\varphi(A)>0$, $A+(n-1)\Delta_0<B$ for some integer n, and

$$\varphi(A)\sinh[(n+1)\cosh^{-1}(2p)^{-1}] -\Phi_0 p\sinh[n\cosh^{-1}(2p)^{-1}]>0,$$

where $p^2=M_0\Delta_0$. Let $A_k=A+k\Delta_0$ ($k=0,\cdots,n-1$), $A_n=\min\{A+n\Delta_0, B\}$, $y(x)\leqq\varphi(A)$ for $A\leqq x<b$; then

$$y(x)\geqq 2p^{k+1}(1-4p^2)^{-\frac{1}{2}}\{\varphi(A)\sinh[(k+1)\cosh^{-1}(2p)^{-1}] -\Phi_0 p\sinh[k\cosh^{-1}(2p)^{-1}]\}$$

for $A_{k-1}\leqq x<A_k$ and $k=1,\cdots,n$. Write

$$\omega=2[1+(1-4p^2)^{\frac{1}{2}}]^{-1},\quad \Omega=2[1-(1-4p^2)^{\frac{1}{2}}]^{-1}.$$

If $\varphi(A)>0$, $B<\infty$, $y(x)\leqq\varphi(A)$, $A\leqq x<b$, and $\varphi(A)>p^2\omega\Phi_0$, then $y(x)>0$, $A\leqq x<B$, and $\lim y(x)>0$ as $x\to B$. If $\varphi(A)>0$, $B=\infty$, $y(x)\leqq\varphi(A)$, $A\leqq x<b$, and $\varphi(A)>p^2\omega\Phi_0$, then $y(x)>C\exp(-\nu x)$ [$\nu=\Delta_0^{-1}\ln\omega$, some $C>0$; $A\leqq x<\infty$]. If $B=\infty$, $y(x)$ is said to be d.s. (diminishing slowly) if $|y(x)|>C\exp(-\nu x)$ for some $C>0$ and x sufficiently large, and is said to be d.r. (diminishing rapidly), if $|y(x)|<C\exp(-\nu' x)$ ($\nu'=\Delta_0^{-1}\ln\Omega$) for some $C>0$ and for all x. If $B=\infty$, then $y(x)$ is d.s. or is d.r. Let $\varphi(x)\geqq 0$, $b<B$ and $y(x)\not\equiv 0$ for $x>b$; if $B<\infty$, then $\lim_{x\to B}y(x)<0$; if $B=\infty$, then $y(x)$ is d.s. *W. J. Trjitzinsky.*

✱Brownell, F. H., 3rd. Non-linear delay differential equations. Contributions to the Theory of Nonlinear Oscillations, pp. 89–148. Annals of Mathematics Studies, no. 20. Princeton University Press, Princeton, N. J., 1950. $4.00.

Let $L(x)$ denote $\sum a_{k,p}x^{(k)}(t-b_p)$ where k is summed from 0 to n and p from 0 to r. The $a_{k,p}$ are constant as are the b_p, while k denotes the order of differentiation. Let $Q(x)$ denote a power series in the terms $x^{(k)}(t-b_p)$ where $0\leqq k\leqq n-1$ and where the terms of Q of degree zero and one are absent. The equation $L(x)+Q(x)=0$ is considered first where the equations $L(x)=0$ have only exponentially decaying solutions, and then without this restriction. In the latter case only periodic solutions are sought for, and in this case the problem is recast as a nonlinear integral equation. The methods of Schmidt are then used. *N. Levinson.*

Dantinne, N. Application de la méthode des approximations successives à l'intégration d'une équation différentielle aux différences. I. Bull. Soc. Roy. Sci. Liége 18, 363–374 (1949).

Dantinne, N. Application de la méthode des approximations successives à l'intégration d'une équation différentielle aux différences. II. Bull. Soc. Roy. Sci. Liége 18, 445–461 (1949).

Consider the differential-difference equation

$$y^{(n)}(x-a_m)+\sum_{i=1}^{n}\sum_{r=1}^{m}A_{n-i,r}(x)y^{(n-i)}(x-a_r)=v(x),$$

where $0=a_1<a_2<\cdots<a_m<1/Ke$, K being a constant dependent upon the upper bounds of the functions $A_{ij}(x)$, $v(x)$, which are assumed to be continuous in $[0,\infty]$. Under these assumptions, it is proved that there exists a solution y for which $y^{(i)}(0)=y_i$, $i=0,1,\cdots,n-1$. The method is that of successive approximations. The awkward assumption $a_m<1/Ke$ arises from the necessity of ensuring the convergence of the Bruwier series $\sum_{n=0}^{\infty}K^n(x+nam)^n/n!$. The presentation could be considerably simplified by the use of vector-matrix notation which would reduce the original equation to one of the form

$$\frac{d}{dt}u(t-a_m)=\sum_{i=1}^{m}B_i(t)u(t-a_i),$$

where u is an n-dimensional column vector and $B_i(t)$ is an $n\times n$ matrix. *R. Bellman* (Stanford University, Calif.).

Kneser, Hellmuth. Reelle analytische Lösungen der Gleichung $\varphi(\varphi(x))=e^x$ und verwandter Funktionalgleichungen. J. Reine Angew. Math. 187, 56–67 (1949).

L'auteur montre d'abord que la résolution de l'équation $\varphi(\varphi(x))=f(x)$, où $f(x)$ est donnée et φ la fonction inconnue, est $\varphi(x)=\psi^{-1}(\psi(x)+\frac{1}{2}\beta)$ si $\psi(x)$ est une solution de l'équation d'Abel $\psi(f(x))=\psi(x)+\beta$ où f et la constante β sont donnés; la fonction inverse ψ^{-1} devant être convenablement précisée. La résolution de l'équation d'Abel étant liée à celle de l'équation de Schröder $\chi(f(x))=\gamma\chi(x)$, γ constante, l'auteur donne d'après Koenigs [Ann. Sci. École Norm. Sup. (3) 1, S.3–S.41 (1884)] la démonstration de l'existence de la solution de l'équation en question mais en affaiblissant les hypothèses de Koenigs: il se place au voisinage d'un point fixe c, $f(c)=c$, et suppose seulement que

$$|f(x)-c-a(x-c)|<M|x-c|^{\delta},$$

M et a sont fixes, $0<|a|<1$, et δ fixe supérieur à 1. La solution est alors obtenue comme dans le cas analytique; elle admet au point c une dérivée égale à 1. Dans le cas analytique, on retrouve le résultat de Koenigs. Pour appliquer ses résultats au cas de $f(x)=e^x$, l'auteur détermine les points fixes de la substitution (x,e^x) et les multiplicateurs correspondants. On est ici dans le cas analytique, le prolongement des solutions et leurs branches réelles sont étudiés. *G. Valiron* (Paris).

Integral Equations

Richard, Ubaldo. Rapporti tra le equazioni di Volterra e le serie di polinomi di Laguerre. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 83, 28–42 (1949).

Discussing the Laguerre expansion of the solution $\varphi(x)$ of Volterra's integral equation

$$(1)\qquad \varphi(x)-\int_0^x K(x-t)\varphi(t)dt=f(x),$$

where $f(x)\sim\sum_0^\infty a_nL_n(x)$ and $K(u)\sim\sum_0^\infty b_nL_n(u)$ with $n!L_n(x)=e^x[x^ne^{-x}]^{(n)}$, the author proves the uniform convergence of this expansion under certain sufficient conditions the study of which is the chief purpose of this paper. The coefficients c_n of the solution (2) $\varphi(x)=\sum_0^\infty c_nL_n(x)$ then can be found with the aid of recurrence equations

$$(1-b_0)c_n=a_n+\sum_{0}^{n-1}c_k(b_{n-k}-b_{n-k-1}).$$

The author's method seems to have practical value in so far as a quick computation of solutions of (1) is concerned.

E. Kogbetliantz (New York, N. Y.).

Bückner, Hans. Ein unbeschränkt anwendbares Iterationsverfahren für Fredholmsche Integralgleichungen. Math. Nachr. **2**, 304–313 (1949).

Es handelt sich um die Integralgleichung

$$(1)\qquad y(s)=f(s)+\lambda\int_a^b K(s,t)y(t)dt,$$

$K(s,t)$ stetig für $a\leqq s,t\leqq b$; $f(t)$ stetig für $a\leqq s\leqq b$ (λ, $K(s,t)$ und $f(s)$ dürfen komplex sein). Es sei $\theta_1,\theta_2,\cdots,\theta_n,\cdots$ eine unendliche Folge von reellen oder komplexen Zahlen mit $\theta_k\neq 1$; ausserdem wird die Existenz einer natürlichen Zahl ρ verlangt, für die $\theta_{k+\rho}=\theta_k$, $k=1,2,\cdots$. Von zwei willkürlichen stetigen Funktionen $u_0(s)$ und $v_0(s)$ ausgehend, definiert der Verf. zwei Folgen von Funktionen $u_0,u_1,u_2,\cdots$; $v_0,v_1,v_2,\cdots$ gemäss den Formeln

$$u_{n+1}(s)=\theta_{n+1}u_n(s)+(1-\theta_{n+1})\lambda\int_a^b K(s,t)u_n(t)dt+(1-\theta_{n+1})f(s),$$

$$v_{n+1}(s)=\theta_{n+1}v_n(s)+(1-\theta_{n+1})\lambda\int_a^b K(s,t)v_n(t)dt.$$

Bei gegebener Parameterfolge θ_n und bei gegebenem λ werden die Folgen $u_n(s)$ und $v_n(s)$ total konvergent genannt, wenn sie für jedes $u_0(s)$ (bzw. $v(s)$) und $f(s)$ konvergieren. Es wird gezeigt, dass man die Parameter $\theta_1,\theta_2,\cdots,\theta_\rho$ stets so wählen kann, dass die Folge $u_n(s)$ gegen die Lösung der Integralgleichung (1) konvergiert, wie auch $u_0(s)$ gewählt sein mag, wenn λ kein Eigenwert ist. Weiter wird gezeigt, dass unter bestimmten Voraussetzungen über die Eigenfunktionen des Eigenwertes λ_k die Grössen $v_{m\rho\pm i}$, $i=1,2,\cdots,\rho-1$, mit $m\rightarrow\infty$ gegen Vielfache einer von $v_0(s)$ abhängigen Eigenfunktion konvergieren. Anwendung auf:

$$y(s)=\lambda\int_a^b K(s,t)y(t)dt,$$

mit $K(s,t)=s(1-t)$ für $0\leqq s\leqq t$ und $K(s,t)=t(1-s)$ für $0\leqq t\leqq s$.

S. C. van Veen (Delft).

Colombo, Giuseppe. Un teorema sulle forme quadratiche e sui nuclei definiti. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 52–59 (1949).

The object of this note is to give conditions on a kernel $H(x,y)$ which is continuous and symmetric in the square $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, in order to make the kernel semi-definite. These sufficient conditions depend on the positiveness of the kernel on the boundary of the square and also on the behavior of the kernel in the two right triangles formed by a diagonal of the square. The author first proves a lemma concerning quadratic forms in a finite number of variables and then uses a limiting process to obtain his result.

I. A. Barnett (Cincinnati, Ohio).

Davison, B. Critical radius of a nearly spherical body in an infinite container. National Research Council of Canada. Atomic Energy Project. Division of Research. **CRT-361** (N.R.C. no. 1820), 64 pp. (1948).

The mathematical problem which is considered in this paper is the solution of the integral equation

$$q(\mathbf{r})=\frac{1}{4\pi}\int\!\!\int\!\!\int[1-\alpha(\mathbf{r}')]q(\mathbf{r}')\frac{e^{-|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|^2}d\mathbf{r}'+\frac{1}{4\pi}\int\!\!\int S(\mathbf{r}'/r')e^{-R_0}R_0^{-2}d\Omega',$$

where $\alpha(r)=\alpha_i$ for $r<a$ and $=\alpha_e$ for $r>a$ (α_i, α_e and a being known constants), $R_0=|\mathbf{r}-a\mathbf{r}'/r|$ is the distance between the point $\mathbf{r}$ and the projection of $\mathbf{r}'$ upon the surface of the sphere $\mathbf{r}'=a$ taken from the center of the sphere,

$$(*)\qquad S(\mathbf{r}/r)=\sum_{l=0}^{\infty}S_l(\mathbf{r}/r)=\sum_{l=0}^{\infty}\sum_{i=-l}^{+l}A_{li}P_l(u_i),$$

$u_i=(x\gamma_{xi}+y\gamma_{yi}+z\gamma_{zi})/r$ and A_{li} are constants. In (*) S_l, which is an arbitrary spherical harmonic of order l, has been expressed as a series in Legendre polynomials P_l for $(2l+1)$ suitably chosen directions γ_{xi}, γ_{yi} and γ_{zi}. [It has been shown earlier by the author that such a representation is always possible; cf. B. Davison, National Research Council of Canada, Division of Atomic Energy, Document no. **MT-124** (N.R.C. no. 1552) (1945); these Rev. **10**, 45.] Further, in the integral equation for $q(\mathbf{r})$ the first integral on the right-hand side is over all space while the second is over all directions $d\Omega'$ for fixed $\mathbf{r}'$.

Expanding $q(\mathbf{r})$ in the form

$$q(\mathbf{r})=\sum_{l=0}^{\infty}\sum_{i=-l}^{+l}A_{li}q_l(r)P_l(u_i)=\sum_{l=0}^{\infty}S_l(\mathbf{r}/r)q_l(r),$$

the author shows that q_l satisfies the integral equation

$$q_l(r)=\int_0^\infty[1-\alpha(r')]q_l(r')F_l(r,r')r'^2dr'+F_l(r,a),$$

where

$$F_l(r,r')=(rr')^{-\frac{1}{2}}\int_1^\infty K_{l+\frac{1}{2}}(kr)I_{l+\frac{1}{2}}(kr')dk\qquad(r>r')$$

$$=(rr')^{-\frac{1}{2}}\int_1^\infty I_{l+\frac{1}{2}}(kr)K_{l+\frac{1}{2}}(kr')dk\qquad(r>r'),$$

and $I_{l+\frac{1}{2}}$ and $K_{l+\frac{1}{2}}$ are the standard Bessel functions for the purely imaginary argument. It is next shown that in the vicinity of a the asymptotic behaviour of q_l is

$$q_l(r)=-\frac{1}{2a^2}\log|r-a|+Q_l+O[(r-a)\log^2(r-a)],$$

where Q_l's are constants. A considerable section of the paper is devoted to the further specification of the constants Q_l. It is shown in particular that

$$Q_l-Q_1\simeq\frac{1}{2a^2}\left[\log l+\frac{l-1}{l}(K-1+\gamma)\right],$$

where γ is Euler's constant and K is the linear term in the expansion of the solution $n_0(r)$ near $r=a$ of the equation

$$n_0(r)=\int_0^\infty[1-\alpha(r')]n_0(r')F_0(r,r')r'^2dr'.$$

It is further established that the functions q_l satisfy the recurrence relation

$$\frac{1}{r^{l+2}}\frac{d}{dr}(r^{l+2}q_{l+1})=r^{l-1}\frac{d}{dr}\left(\frac{q_{l-1}}{r^{l-1}}\right)+(2l+1)q_l\frac{a^{l-1}+(\alpha_e-\alpha_i)\int_0^a r^{l+1}q_{l-1}dr}{a^l+(\alpha_e-\alpha_i)\int_0^a r^{l+2}q_l dr}.$$

Accordingly, all the functions q_l can be determined successively in terms of q_0 and q_1. From the equations satisfied by

$q_1(r)$ and $n_0(r)$ it readily follows that

$$q_1(r) = -\frac{1}{(\alpha_s-\alpha_i)a^2n_0(r)}\frac{dn_0(r)}{dr}.$$

The determination of q_2 requires some further consideration. Finally it is shown that

$$\int_0^\infty q_1(r)F_1(r,a)[1-\alpha(r)]r^2dr$$

$$=\max\frac{\left[\int_0^\infty \bar{q}_1(r)F_1(r,a)[1-\alpha(r)]r^2dr\right]^2}{\int_0^\infty \bar{q}_1(r)\left[\bar{q}_1(r)-\int_0^\infty \bar{q}_1(r')\times[1-\alpha(r')]F_1(r,r')r'^2dr'\right][1-\alpha(r)]r^2dr},$$

where $\bar{q}_1(r)$ is unrestricted provided it leads to convergent integrals. This relation is made the basis of a variation calculation of the functions $q_i(r)$. *S. Chandrasekhar.*

LeCaine, Jeanne. The neutron density near a plane surface in a capturing medium. Canadian J. Research. Sect. A. 28, 242–267 (1950).

The author summarizes his paper as follows. "The variational method is applied to the Milne problem with capture, to obtain, for the density and for the angular distribution, expressions which give quite accurate values and which have a relatively simple analytical form. Tables of the density and angular distribution are included. The extrapolation distances determined by the variational method agree to high accuracy with the exact values. The angular distribution determined by the variational method is shown to be accurate within 0.05% in all cases where exact results are available. Tables are also given for the density and angular distribution in the Milne problem with capture and constant production." *A. Heins* (Pittsburgh, Pa.).

Rosenthal, Jenny E. The solution of a certain general type of integral equation. Proc. Nat. Acad. Sci. U. S. A. 36, 267 (1950).

It is shown that the equation $\int_0^1[f''(1)f(u)+2f'^2(u)]du=0$ has solutions $f(u)=Au+F(u)$, where A satisfies a quadratic equation and $F(u)$ is any analytic function. A generalization is indicated. *W. J. Trjitzinsky* (Urbana, Ill.).

Sobolev, V. I. On a nonlinear integral equation. Doklady Akad. Nauk SSSR (N.S.) 71, 831–834 (1950). (Russian)

The author considers the equation

$$(*)\qquad \int_B K(s,t)g(t,x(t))dt=\mu x(s)$$

(B a bounded domain in n-space), where $K(s,t)$ is a symmetric, positive, continuous kernel and $g(t,u)$ is measurable for t in B and for all real u, and satisfies

$$(1)\qquad |g(t,u_1)-g(t,u_2)|\leq N|u_1-u_2|,$$

(2) $g(t,-u)=g(t,u)$, (3) $ug(t,u)>0$ for $u\neq 0$. The author uses variational-topological methods, first introduced by Lusternik [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 3, 257–264 (1939)]. Let $\{\varphi_i(s)\}$, $\{\lambda_i\}$, $0<\lambda_i\leq\lambda_{i+1}$, be complete systems of characteristic functions and numbers of $K(s,t)$ with $\sum\lambda_i^{-1}<\infty$. The set of elements $x=\{\xi_i\}$, where ξ_i are real and are such that $\sum\lambda_i\xi_i^2<\infty$, is denoted by H. On letting $(x,y)=\sum\lambda_i\xi_i\eta_i$, $\|x\|^2=\sum\lambda_i\xi_i^2$ and H is a real, separable Hilbert space. If $x=\{\xi_i\}$ is in H, then $x(s)=\sum\xi_i\varphi_i(s)$ is in $L_2(B)$. A functional

$$f(x)=\int_B G(t,\sum\xi_j\varphi_j(t))dt,$$

where $G(t,u)=\int_0^u g(t,v)dv$, is considered in an arbitrary sphere $\sum\lambda_i\xi_i^2\leq C^2$. Then $f(x)$ is differentiable in the sense of Fréchet and $df(x,h)=\sum g_i(\xi)h_i$, where

$$g_i(\xi)=\int_B g(t,\sum\xi_j\varphi_j(t))\varphi_i(t)dt.$$

The correspondence between elements $x\{\xi_i\}$, $y=\{\lambda_i^{-1}g_i(\xi)\}$ is expressed by the operator $Ax=y$; then $df(x,h)=(Ax,h)$ and A is a symmetric operator generated by $f(x)$. It is shown that A is completely continuous, positive, satisfies a Lipschitz condition, and $A(-x)=-A(x)$. With the aid of an earlier paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) 31, 735–737 (1941); these Rev. 3, 208] it is proved that the equation (*) has at least a denumerable infinity of distinct real characteristic functions $x_n(s)=\sum\xi_i^{(n)}\varphi_i(s)$ and characteristic numbers μ_n; also $\sum\lambda_i(\xi_i^{(n)})^2=C^2$.

W. J. Trjitzinsky (Urbana, Ill.).

Cameron, R. H., and Martin, W. T. Non-linear integral equations. Ann. of Math. (2) 51, 629–642 (1950).

The authors combine the results of a previous paper on the orthogonal development of nonlinear functionals [Ann. of Math. (2) 48, 385–392 (1947); these Rev. 8, 523] with those of a recent paper on the transformation of Wiener integrals by nonlinear transformations [Trans. Amer. Math. Soc. 66, 253–283 (1949); these Rev. 11, 116] in order to obtain explicit solutions for a class of nonlinear functional equations. More precisely, let T be a one-to-one transformation of the Wiener space C onto itself,

$$(1)\qquad T: y(t)=x(t)+\Lambda(x|t)\equiv G(x|t),$$

let its first variation

$$\delta\Lambda(x|t|z)\equiv\frac{\partial}{\partial v}\Lambda(x+vz|t)|_{v=0}$$

exist for all suitably restricted (x,t,z), and let it be representable in the form

$$\delta\Lambda(x|t|z)=\int_0^1 K(x|t,s)z(s)ds.$$

Then, with appropriate smoothness and auxiliary conditions on Λ and K, the solution $x(t)$ of (1) is given by

$$(2)\qquad x(t)=\text{l.i.m.}\sum_{n\to\infty\; m_1,\dots,m_n=0}^{n} A_{m_1,\dots,m_n}(t)\Psi_{m_1,\dots,m_n}(y),$$

where the Ψ's are the Fourier-Hermite polynomials discussed by the authors in the first reference given above, the l.i.m. is taken as y ranges over C, and the coefficients are given by

$$A_{m_1,\dots,m_n}(t)=\int_C^w z(t)\Psi_{m_1,\dots,m_n}[G(z|\cdot)]J_1(z)d_wz,$$

where $J_1(z)$ is a certain functional of Λ. Under certain additional conditions, (2) holds with the l.i.m. interpreted in the overall sense as (y,t) ranges over $C\otimes[0,1]$. Under still more stringent conditions, the series in (2) can be replaced by a series convergent for each $y\epsilon C$. Applications are made to the Volterra and Fredholm integral equations of the second kind by choosing $\Lambda(x|t)$ to be $\int_0^t F(t,s,x(s))ds$ and $\int_0^1 F(t,s,x(s))ds$,

respectively. It should be mentioned that the function $x(t)$ in eq. (4.10), p. 634, should be replaced by $z(t)$.
R. E. Graves (Minneapolis, Minn.).

Functional Analysis

★Rey Pastor, Julio. Functional analysis and the general theory of functions. Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 339–372, Rome, 1943. (Spanish)

This is entirely a bibliographical commentary on some of the literature of functional analysis, with a few introductory general observations and comments. Approximately 400 titles are listed. The period covered is about 1900–1940, and one observes that the work was written a number of years ago. There are five parts: (I) Three methods of functional analysis (partly typified by Pincherle, Volterra, and Frechet, respectively). (II) Functional spaces. (III) Linear functionals. (IV) Representations of linear functionals. (V) Applications of functional analysis. *A. E. Taylor.*

Wielandt, Helmut. Zur Abgrenzung der selbstadjungierten Eigenwertaufgaben. I. Räume endlicher Dimension. Math. Nachr. 2, 328–339 (1949).

For a finite dimensional complex linear space $\mathfrak{R}$ with inner product (x, y), the author calls a linear subspace $\mathfrak{P}$ of the product space $\mathfrak{R}\oplus\mathfrak{R}$ a coupling (Paarung) in $\mathfrak{R}$; an element in $\mathfrak{P}$ of the form $\{\lambda'e, \lambda''e\}$, with e a nonzero element of $\mathfrak{R}$ and λ', λ'' complex numbers not both zero, is termed a proper (eigen-) solution of $\mathfrak{P}$ corresponding to the proper (eigen-) value $\lambda=\lambda''/\lambda'$ ($\lambda=\infty$ if $\lambda'=0$). Corresponding to the usual introduction of the adjoint operator in Hilbert space, the coupling $\mathfrak{P}^*$ adjoint to $\mathfrak{P}$ is defined as the totality of pairs $\{u, v\}$ such that $(u, y)=(v, x)$ for each $\{x, y\}$ of $\mathfrak{P}$; the product $\mathfrak{Q}\mathfrak{P}$ of couplings $\mathfrak{P}, \mathfrak{Q}$ is defined to be the totality of pairs $\{x, y\}$ for which there is a z such that $\{x, z\}\epsilon\mathfrak{P}$ and $\{z, y\}\epsilon\mathfrak{Q}$. The principal results of the paper are concerned with the characterization of couplings that are self-adjoint $[\mathfrak{P}=\mathfrak{P}^*]$, or normal $[\mathfrak{P}\mathfrak{P}^*=\mathfrak{P}^*\mathfrak{P}]$, with special attention to the matrix problems $Ax=\lambda Bx$ in the column-space of n-tuples x of complex numbers, associated with the coupling consisting of all pairs $\{x, y\}$ satisfying $Ax=By$. *W. T. Reid* (Evanston, Ill.).

Gelbaum, Bernard R. Expansions in Banach spaces. Duke Math. J. 17, 187–196 (1950).

The paper is mainly concerned with certain notions about bases in Banach spaces. If E is a Banach space, and if $\{x_n\}$ is a basis for E, so that each $x\epsilon E$ has a unique expansion $x=\sum_1^\infty a_nx_n$, the coefficients in the expansion define elements $X_n\epsilon E^*$ such that $X_n(x)=a_n$. We refer to the system $\{x_n; X_n\}$ as a basis for E. If $\sum_1^\infty a_nx_n$ is always unconditionally convergent to x, the basis is called absolute. It will be recalled that there is a "canonical" imbedding of E in E^{**}, and that E is reflexive if and only if the canonical image $J(E)$ of E in E^{**} is all of E^{**}. A basis $\{X_n; f_n\}$ for E^* is called a retro-basis if all the elements $f_n\epsilon E^{**}$ lie in $J(E)$. Examples of absolute, nonabsolute, retro- and nonretro-bases are given. A retro-basis need not be absolute, and vice versa. Retro-bases for E^* are characterized by the following theorem. If $\{X_n\}$ is a basis for E^*, let M_k denote the closed linear manifold determined by all the X_n with $n\neq k$. Then $\{X_n\}$ is a retro-basis if and only if each M_k is regularly closed. There are various theorems about complemented manifolds, projections, and various kinds of bases. There are also some theorems about reflexivity. There is a brief discussion of absolute bases in Hilbert space, using the Rademacher functions. The paper concludes with some theorems about T-bases. If T is a Toeplitz-matrix, a set $\{x_n\}$ is a T-basis if every $x\epsilon E$ has a unique formal expansion $\sum_1^\infty a_nx_n$ which is T-summable to x. *A. E. Taylor* (Los Angeles, Calif.).

Dieudonné, J. Deux exemples singuliers d'équations différentielles. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 38–40 (1950).

The author considers the differential equation

$$\mathbf{x}'=\mathbf{f}(t, \mathbf{x}), \tag{1}$$

where t is a real variable and $\mathbf{x}$ is a vector in a Banach space E over the reals. He takes two theorems concerning solutions of (1) which are valid for E finite-dimensional and shows, by examples, that they can fail for E infinite-dimensional. The first is that of Peano which states that if $\mathbf{f}$ is continuous in the neighborhood of $(t_0, \mathbf{x}_0)$, there is at least one solution of (1) defined in a neighborhood of t_0 which has the value $\mathbf{x}_0$ at t_0. The second is concerned with the behavior of a solution $\mathbf{u}(t)$, $\mathbf{u}(t_0)=\mathbf{x}_0$ at the boundary of the largest interval emanating from t_0, which satisfies (1). In both counter-examples, E is taken as (c_0). *B. Yood*

Graves, Lawrence M. Some mapping theorems. Duke Math. J. 17, 111–114 (1950).

In part I the author proves that, if a continuous function $G(x)$ from one Banach space $(\mathfrak{X})$ into another $(\mathfrak{Y})$, defined for $\|x\|<\gamma$ and vanishing at the origin, can be approximated sufficiently closely by a linear continuous transformation taking $\mathfrak{X}$ into the whole of $\mathfrak{Y}$, then $G(x)$ takes $\|x\|<\gamma$ into a neighbourhood of the origin in $\mathfrak{Y}$. He deduces a theorem on functions of class C' [by which he apparently means a function whose Fréchet differential $dG(x; dx)$ is a continuous function of x] and a theorem on linear continuous transformations. In part II he derives similar results when $G(x)$ is defined only on part of a cone near its vertex, and a solution of $y=G(x)$ is sought only for y in a compact piece of a cone in $\mathfrak{Y}$. *A. F. Ruston* (London).

Rickart, C. E. Isomorphic groups of linear transformations. Amer. J. Math. 72, 451–464 (1950).

The first principal theorem of the author is a generalization of the following result of the reviewer [Ann. of Math. (2) 43, 224–260 (1942); Trans. Amer. Math. Soc. 57, 155–207 (1945); these Rev. 4, 12; 6, 274]. Let X_L and Y_M be real linear systems; that is, let X and Y be real linear spaces and let L (M) be a total subspace of the set of all linear functionals on X (Y). Let the group G_1 of all automorphisms of X_L (as a linear system) be isomorphic as an abstract group to the corresponding group for Y_M. Then either X_L and Y_M are isomorphic linear systems or X_L and M_Y are isomorphic linear systems. The author's generalization consists first in replacing the real field by an arbitrary division ring of characteristic different from two, and second in replacing the group of all automorphisms by any subgroup which contains all one dimensional involutions. A priori the division rings for the two systems are not assumed to be isomorphic. The other principal theorem of the paper asserts that the isomorphism between the groups G_1 and G_2 of the two systems differs from that defined by the vector space isomorphism only in being combined with a homo-

morphism of G_1 into the multiplicative group of the division ring. This theorem in the finite-dimensional case is due to Dieudonné [C. R. Acad. Sci. Paris **225**, 914–915 (1947); these Rev. **9**, 494]. In this case it is an extension of a theorem of Schreier and van der Waerden [Abh. Math. Sem. Univ. Hamburg **6**, 303–322 (1928)].

G. W. Mackey (Cambridge, Mass.).

Calculus of Variations

Nöbeling, G. Über die erste Randwertaufgabe bei regulären Variationsproblemen. I. Math. Z. **51**, 712–751 (1949).

The present paper is concerned with existence theorems for surfaces minimizing an integral of the form

$$I(h, G)=\int\int_G F(h_x, h_y)dx\,dy,$$

when $F(p,q)$ is analytic on a domain $p^2+q^2<D^2$, $0<D\leqq+\infty$, and $F_{pp}>0$, $F_{pp}F_{qq}-F^2_{pq}>0$. By an admissible curve R is meant one which (1) is composed of a finite number of disjoint simply closed curves, (2) has a simple projection K in the (x, y)-plane, (3) is the boundary of an open connected set $G(R)$, (4) has associated with it a number r, $0<r\leqq+\infty$, such that through each point P of K there passes a circle of radius r which together with its interior is in the complement of $G(R)$ and (5) there exist numbers c, d with $0<c\leqq+\infty$ and $0<d<D$ such that through each point P of R there exist two spheres of radius c and gradient $\leqq d$ at P which are tangent to R at P, one from above and the other from below. The main results are the following. Given r_0, c_0, d_0 $(0<r_0\leqq+\infty, 0<c_0\leqq+\infty, 0<d_0<D)$ there exists a number $\zeta_0>0$ such that for each admissible curve R with $r\geqq r_0$, $c\geqq c_0$, $d\leqq d_0$ and diameter $<\zeta_0$, there exists a surface minimizing $I(h, G)$ within the given boundary R. In particular, if $c=+\infty$, a solution exists.

M. R. Hestenes.

Nöbeling, G. Über die erste Randwertaufgabe bei regulären Variationsproblemen. II. Math. Z. **52**, 1–31 (1949).

Let $\mathfrak{R}$ be a class of admissible arcs R [see the preceding review] with a metric Δ such that to each arc R_0 in $\mathfrak{R}$ there is an $\epsilon_0>0$ such that every arc R in $\mathfrak{R}$ with $\Delta(R_0, R)<\epsilon_0$ satisfies properties (4) and (5) with the same numbers r, c, d. It is shown that the class $\mathfrak{R}_1$ of $\mathfrak{R}$ for which the minimum problem has a solution is open. If we denote by $d(R)$ the maximum of the gradient of a solution $H(x, y)$ with boundary R, and if $\{R_i\}$ is a sequence of curves in $\mathfrak{R}_1$ converging to a curve in $\mathfrak{R}-\mathfrak{R}_1$, then $d(R_i)\to D$.

M. R. Hestenes.

***Boerner, Hermann. Variationsrechnung.** Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 53–65. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10=\$2.40.

The present paper is a review of the work in the calculus of variations done by various authors during the period 1941–1945. The authors are W. Damköhler, E. Hopf, H. R. Weber, E. Hölder, B. Manià, J. Radon, W. Threlfall, G. Nöbeling, N. Sakellariou, H. Kneser, H. Gericke, H. Boerner, G. Bol, A. Dinghas, E. Reiche, Th. Vahlen, and E. Schmidt. For the papers of G. Nöbeling see the two preceding reviews.

M. R. Hestenes (Los Angeles, Calif.).

Viola, Tullio. L'integrale in forma ordinaria, alla frontiera d'un campo dello spazio funzionale lagrangiano del prim'ordine. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 673–679 (1949).

This note is an abstract of another paper [Ann. Scuola Norm. Super. Pisa (3) **1**, 101–160 (1949); these Rev. **11**, 526].

L. M. Graves (Chicago, Ill.).

Viola, Tullio. Un problema metrico relativo agli insiemi di punti, nel piano o nello spazio. Boll. Un. Mat. Ital. (3) **5**, 64–67 (1950).

Given a connected open plane set C, its closure $\bar{C}$, and two points $P_1, P_2\epsilon C$, the problem of determining and studying (a) the rectifiable curves Γ of minimum length joining P_1 and P_2 and belonging to $\bar{C}$ has some variants, namely, (b) the curves $\Gamma_1\subset\bar{C}$ which are limits of minimizing sequences of curves $\gamma\subset C$, and (c) the curves $\Gamma_2\subset\bar{C}$ which can be obtained from curves $\gamma\subset C$ by continuous deformations reducing the length. Elementary statements for the problems (b) and (c) are given in preparation for the discussion (in following papers) of the analogous problems in Euclidean 3-space.

L. Cesari (Lafayette, Ind.).

Castoldi, Luigi. Sopra una formulazione della condizione variazionale di "trasversalità" utile nella fisica matematica. Atti Accad. Ligure **5**, 111–120 (1949).

The author considers the variational problem

$$\delta\int\int F(x, y, z, z_x, z_y)dxdy=0,$$

the domain of integration being bounded by the curve Γ of intersection of the surface S with equation $z=z(x, y)$ and a given surface $\phi(x, y, z)=0$. Introducing parameters u, v, he obtains three partial differential equations satisfied by x, y, z on S and three transversality conditions on Γ. He proceeds to give two other equivalent forms for the transversality conditions and illustrates the results by an application to the theory of capillarity.

J. L. Synge.

Van Hove, Léon. Sur le signe de la variation seconde des intégrales multiples à plusieurs fonctions inconnues. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **24**, no. 5, 68 pp. (1949).

In his first two chapters the author reviews known results concerning the conditions of Legendre and Jacobi and the positiveness of the second variation for multiple integrals

$$\int F(x, z, p)dx$$
$$=\int F(x_1, \cdots, x_\mu, z_1, \cdots, z_n, p_{11}, \cdots, p_{n\mu})dx_1\cdots dx_\mu$$

where $p_{j\alpha}=\partial z_j/\partial x_\alpha$. For a proof of the sufficiency of the Legendre condition for a "local minimum" for general values of μ and n, he refers to an earlier paper by himself [Nederl. Akad. Wetensch., Proc. **50**, 18–23 = Indagationes Math. **9**, 3–8 (1947); these Rev. **8**, 522]. He remarks that when the second variation has the special form $\delta^2 I=\int F_{p_{j\alpha}p_{k\beta}}\zeta_{j\alpha}\zeta_{k\beta}dx$ where $\zeta_{j\alpha}=\partial\zeta_j/\partial x_\alpha$, and the Legendre condition in the strict sense is satisfied, then the integral I has a weak minimum (for fixed boundary values), when either $n=1$ or $\mu=1$, but not necessarily when $n>1$ and $\mu>1$. In chapter 3 the spectral theory for linear continuous self-adjoint operators in real separable Hilbert space is reviewed, and in chapter 4 this theory is applied to obtain a condition for the positiveness of the second variation, expressed in terms of a second

operator which may be chosen arbitrarily subject to the restriction that it be positive definite. In chapter 5 it is shown how this second operator may be chosen so as to lead to the previously known conditions when either $n=1$ or 2, or $\mu=1$ or 2. Chapter 6 sketches the relations of the results with the theory of self-adjoint systems of linear partial differential equations of elliptic type, and gives an application to a problem of elastic deformation. The references to other work on the subject are few, and indicate that the author may not be familiar with some recent papers on multiple integrals, as listed, e.g., in "Contributions to the Calculus of Variations, 1938–1941," University of Chicago Press, 1942 [see these Rev. 4, 46–48]. In particular, a number of ideas the author uses are treated in a memoir of Morrey [Univ. California Publ. Math. (N.S.) 1, 1–130 (1943); these Rev. 6, 180]. The paper contains a number of misprints, including the use in chapter 5 of "absolument continu" for "totalement continu." *L. M. Graves.*

Theory of Probability

Kendall, M. G. On the reconciliation of theories of probability. Biometrika 36, 101–116 (1949).

In this attempt at reconciliation of current theories of the foundations of probability and statistical inference, the author makes the following points. (1) The postulational method is insufficient in probability. In other branches of science it may be enough to set up a mathematical model and accept it if it gives a reasonably good account of observation; but in the most general sense we are here considering how good a "reasonably good" agreement must be. (2) Any theory of probability which does not take probability itself as a primitive idea must, in some form or other, introduce an equivalent primitive before it can be applied. (3) On the other hand, any theory of probability must, if it is to be applicable to reality, assume that propositions with greater probability are true more often, in fact, than propositions with lesser probability. (4) The two main schools, the frequentists and the nonfrequentists, each contain part of the truth, and are defective mainly in their omissions: the former, regarding point (2) and the latter, regarding point (3). But the question is tied up with the problems of the inverse theory, to which the author then turns his discussion, making his remaining points. (5) Methods based on Bayes' theorem have difficulty in arriving at a priori probabilities, but these may perhaps be resolved, since after many trials initial differences in assumed a priori probabilities disappear, and we almost always have general experience from which a priori probabilities may be intuitively inferred. In the few cases where no such experience exists, equal a priori probabilities would seem to be appropriate. (6) The non-a priori methods (maximum likelihood, fiducial inference, confidence intervals) have the practical disadvantage of not providing a place for a priori knowledge, when it exists, a fact which can lead to practically absurd conclusions, if the methods are applied rigidly. (7) Maximum likelihood and fiducial inference have this difficulty of principle: either they do not introduce a new concept, in which case they have to be regarded as mere derivatives of Bayes' theorem; or else they do introduce such a concept, in which case they require a new postulate. Such a postulate would have to contain an empirical assumption similar to that considered under point (3). While the reviewer has always been in agreement with the main line of the author's position, he has difficulty with point (3); is the more frequent occurrence of the more probable event to be regarded as a statement of fact, or of "probability"? Either view leads to fundamental difficulties. And he hesitates to accept the last remark in point (5). *B. O. Koopman* (New York, N. Y.).

Bunický, E. Remarque sur le critère de l'indépendance des caractères. Aktuárské Vědy 8, no. 2, 53–60 (1949).

The author obtains a number of necessary conditions for the independence of two events (characters). It is shown that some of the conditions are sufficient whereas others are not. *A. H. Copeland* (Ann Arbor, Mich.).

Ugaheri, Tadashi. On a limit distribution. Ann. Inst. Statist. Math., Tokyo 1, 157–160 (1950).

Let X_0 have density l^{-1} in $0 \leqq x \leqq l$; if $X_{n-1}=x_{n-1}$, let X_n have density x_{n-1}^{-1} in $l-x_{n-1} \leqq x \leqq l$. The author shows that as $n\to\infty$ the limit of the probability density of X_n exists and is equal to $2l^{-2}x$ in $0 \leqq x \leqq l$. [The proof can be shortened by differentiating (6) to get

$$(l-x)^2 P''_{n+1}(x) = P_n(l-x) - (l-x)P_n'(l-x) = \int_0^{l-x} [P_n'(u) - P_n'(l-x)]du$$

from which it follows that if $P_n''(x) \geqq 0$, then $P''_{n+1}(x) \leqq 0$.] *K. L. Chung* (New York, N. Y.).

Chung, Kai Lai. Fluctuations of sums of independent random variables. Ann. of Math. (2) 51, 697–706 (1950).

Soit $\{X_n\}$ une suite indéfinie de variables aléatoires indépendantes, de même fonction de répartition $F(x)$; posons $S_k = \sum_{n=1}^k X_n$, $\int_{-\infty}^{+\infty} x dF(x) = \alpha$, $\int_{-\infty}^{+\infty} |x|^i dF(x) = \beta_i$, et $\sigma^2 = \beta_2 - \alpha^2$. La fonction $F(x)$ est dite du "type treillis" si elle est totalement discontinue et si ses discontinuités [valeurs de x pour lesquelles $F(x)$ est discontinue] font partie d'une progression arithmétique. L'auteur établit les deux théorèmes suivants où c est un nombre fixe quelconque. (I) Si $F(x)$ n'est pas du type treillis et si $\alpha=0$, $\sigma^2=1$, $\beta_3<\infty$, et si T_n est le nombre des $k \leqq n$ pour lesquels $S_k > c$, $S_{k+1} < c$, on a:

$$(1) \qquad \lim_{n\to+\infty} \Pr(T_n \leqq \tfrac{1}{2}\beta_1 n^{\frac{1}{2}} x) = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}} \int_0^x e^{-y^2/2} dy.$$

(II) Si $F(x)$ est du type treillis, si (ce qui ne diminue pas la généralité) on suppose son "envergure" (plus petite distance de deux de ses discontinuités) égale à 1, si $0 \leqq \alpha < 1$, $\sigma^2 > 0$, $\beta_3 < \infty$, et si T_n est le nombre des $k \leqq n$ pour lesquels $S_k - k\alpha > c$, $S_{k+1} - (k+1)\alpha < c$, on a:

$$(2) \qquad \lim_{n\to+\infty} \Pr(T_n \leqq \gamma n^{\frac{1}{2}} x) = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}} \int_0^x e^{-y^2/2} dy,$$

où $\gamma = \frac{1}{2}[\beta_1 + 2\alpha F(0)]$. *R. Fortet* (Los Angeles, Calif.).

Fortet, R. Quelques travaux récents sur le mouvement brownien. Ann. Inst. H. Poincaré 11, 175–226 (1949).

Expository paper. The first part is concerned principally with the derivation of various distributions connected with the Brownian motion process, leaning heavily on the work of Kac. The second part is concerned principally with the stochastic processes suggested by Brownian motion and noise problems, the multidimensional stationary Gaussian Markov processes, and leans heavily on the work of the reviewer. *J. L. Doob* (Urbana, Ill.).

Roseau, Maurice. Sur une classe de fonctions aléatoires. C. R. Acad. Sci. Paris 230, 1497–1499 (1950).

Let $\{U(x), a \leqq x \leqq b\}$ be a family of random variables (stochastic process) with parameter set $[a, b]$, and let ϕ be the covariance function of the process, $\phi(x, y) = E\{U(x)\overline{U(y)}\}$. Suppose that for each t the process is transformed into a new one, $\{U(x, t), a \leqq x \leqq b\}$, by the formula

$$U(x, t) = S_t U(x) = \int_a^b K(x, y; t) U(y) dy,$$

where $S_{t+u} = S_t S_u$, $S_0 = I$. The author solves the problem of finding kernels K which leave the covariance function invariant in certain special cases. *J. L. Doob.*

Tager, P. G. The frequency spectrum in phase-pulse modulation. Avtomatika i Telemehanika 8, 117–135 (1947). (Russian)

Pulse-phase modulation is a modulation scheme in which a signal $F(t)$ is translated into a sequence of impulses. The signal $F(t)$ is sampled at a definite rate, say P times per second, and the nth sample value $F(nP)$ determines the time t_n of arrival of the nth impulse by means of $t_n = nP + cF(nP)$, where c is a constant. The particular signal with which the paper deals is $F(t) = A + B \sin(\omega t + \beta)$. A Fourier analysis is performed on the pulse-phase modulated version of $F(t)$ assuming the impulses to be of rectangular shape. *E. N. Gilbert* (Murray Hill, N. J.).

Mathematical Statistics

Féron, Robert. De l'information. C. R. Acad. Sci. Paris 230, 1495–1497 (1950).

General remarks on variance, conditional variance, and quantities playing similar roles. *J. L. Doob.*

Gumbel, E. J., and von Schelling, H. The distribution of the number of exceedances. Ann. Math. Statistics 21, 247–262 (1950).

The authors deal with the probability that the mth largest value in a sample of size n will be exceeded X times in a second (future) sample of size N; the random variable X is called the number of exceedences. They determine the means, moments and cumulative distribution of X. For the case in which $n = N = 2k - 1$, and $m = k$, the limiting distribution of $(X-k)/k^{\frac{1}{2}} = Z$, say, as $n \to \infty$, is shown to be normal with mean and variance equal to k. In case N and n are large and m is small, a limiting distribution of X is determined. The authors carry out some computations of means and distributions and exhibit the results graphically. *S. S. Wilks* (Princeton, N. J.).

Moran, P. A. P. The distribution of the multiple correlation coefficient. Proc. Cambridge Philos. Soc. 46, 521–522 (1950).

The distribution of R, the sample multiple correlation coefficient between x_1 and $x_2, \cdots, x_n$, having a normal distribution, is obtained by making use of $1 - R^2 = (1 - r_{12}^2)(1 - S^2)$, where r_{12} is the correlation between x_1 and x_2 and S is the correlation between x_1 and $x_3, \cdots, x_n$, when the effect of x_2 is removed. *T. W. Anderson* (New York, N. Y.).

Krishna Iyer, P. V. The theory of probability distributions of points on a lattice. Ann. Math. Statistics 21, 198–217 (1950).

The following is an extract from the author's summary. "This paper discusses the theory of certain probability distributions arising from points arranged in the form of lattices in two, three, and higher dimensions. The points are of k characters which for convenience are described as colors. A two-dimensional lattice will consist of $m \times n$ points in m columns and n rows. In a three-dimensional lattice there will be $l \times m \times n$ points in the form of a rectangular parallelopiped. Two situations arise for consideration. They are, to use the term of Mahalanobis, free and nonfree sampling. In free sampling the color of each point is determined, on null hypothesis, independently of the color of the other points. The probabilities of the points belonging to the different colors, say black, white, etc., are $p_1, p_2, \cdots, p_k$, such that $\sum_1^k p_r = 1$. In non-free sampling the number of points of each color is specified in advance, say $n_1, n_2, \cdots, n_k$ so that $\sum_1^k n_r = mn$ or lmn according as the lattice is two- or three-dimensional. Only the arrangements of these points in the lattice are varied.

The distributions considered in this paper are the following: (i) the number of joins between adjacent points of the same colors, say black-black joins, (ii) the number of joins between adjacent points of two specified colors, say black-white joins, and (iii) the total number of joins between points of different colors, along mutually perpendicular axes. The methods used here are the same as those developed by the author [J. Indian Soc. Agric. Statistics 1, 173–195 (1948); these Rev. 11, 446] for the linear case. All the distributions tend to the normal form when l, m and n tend to infinity, provided the p's are not very small." *J. Wolfowitz* (New York, N. Y.).

Westenberg, J. A tabulation of the median test for unequal samples. Nederl. Akad. Wetensch., Proc. 53, 77–82 = Indagationes Math. 12, 8–13 (1950).

An extension to unequal sample sizes of a nonparametric test previously given [same Proc. 51, 252–261 (1948); these Rev. 10, 52]. Tables are given for sample sizes 6, 10, 20, 50, 100, 200, 500, 1000, 2000, and for significance levels 5, 4, 3, 2, 1, $\frac{1}{2}$, $\frac{1}{10}$%. *C. P. Winsor* (Baltimore, Md.).

Noether, Gottfried Emanuel. Asymptotic properties of the Wald-Wolfowitz test of randomness. Ann. Math. Statistics 21, 231–246 (1950).

The paper investigates asymptotic properties of the serial correlation R_h, proposed by Wald and Wolfowitz as a test of randomness [Ann. Math. Statistics 14, 378–388 (1943); these Rev. 5, 211]. Previously given conditions for normal tendency of R_h are weakened to the requirement of positive variance and finite absolute moment of order $4+\delta$. Conditions are given for consistency against trend and cyclical movement. Four tests against cyclical movement, R_h and the corresponding quantity based on ranks are compared. The Mann-Whitney T statistic is shown to be asymptotically more efficient than R_h when testing against trend. *J. L. Hodges, Jr.* (Berkeley, Calif.).

Barankin, E. W. Extension of a theorem of Blackwell. Ann. Math. Statistics 21, 280–284 (1950).

For any numerical chance variable x and any chance variable y, $E|x|^s \geqq E|E(x|y)|^s$ for all $s \geqq 1$. Equality holds for $s=1$ if and only if sgn x is a function of y, and for $s>1$ if and only if x is a function of y. Thus, the method given

by the reviewer for improving an unbiased estimate x by replacing it by $E(x|y)$, where y is a sufficient statistic, cannot increase the sth absolute central moment of the estimate for any $s \geqq 1$, and actually diminishes these moments except in the cases specified. *D. Blackwell.*

Sichel, Herbert S. The method of frequency-moments and its application to type VII populations. Biometrika 36, 404–425 (1949).

This is an investigation of the value of the method of frequency moments [cf. Sichel, J. R. Statist. Soc. (N.S.) 110, 337–347 (1947); these Rev. 10, 212] in the estimation of the parameters of frequency distributions. It appears that the method provides reasonably efficient estimates for leptokurtic type VII distributions, for which the usual moment estimates may break down completely. Numerical examples are given. *C. P. Winsor* (Baltimore, Md.).

Haldane, J. B. S. A note on non-normal correlation. Biometrika 36, 467–468 (1949).

The effect of nonnormality on the precision of the product-moment estimate of a correlation coefficient is investigated for a particular case. It appears that skewness is of little effect, but that with high correlations changes in kurtosis may affect the precision of the estimate considerably.

C. P. Winsor (Baltimore, Md.).

Thompson, William R. Use of moving averages and interpolation to estimate median-effective dose. Bacteriological Rev. 11, 115–145 (1947).

The author proposes to estimate doses giving 50% response by moving averages and interpolation. Arguments and examples are given to show that this method is more desirable than others of comparable simplicity [Kärber, Reed-Muench] and less sensitive to assumptions about the underlying distribution than the probit methods of Bliss and Fisher. *C. P. Winsor* (Baltimore, Md.).

Scott, Elizabeth L. Note on consistent estimates of the linear structural relation between two variables. Ann. Math. Statistics 21, 284–288 (1950).

Another case is presented in which the structural linear relation between two observable random variables can be consistently estimated. Let $x=\xi+u$, $y=\alpha+\beta\xi+v$ be observables in which α and β are the parameters to be estimated and ξ, u, and v are independent random variables. Here u and v are assumed to be normally distributed about zero means, but it is known that if ξ is also normally distributed, then α, β and the variances of u and v are unidentifiable. The present paper gives a method of finding a consistent estimate of β, the key to the problem, on the assumption that an odd central moment of ξ exists and is different from 0. The proof is based on a stochastic limit theorem for a sequence of functions which, as has been pointed out by Neyman according to the author, can be used as a basis for an elementary proof of the consistency of maximum likelihood estimates. *C. C. Craig* (Ann Arbor, Mich.).

Hamaker, H. C. The theory of sampling inspection plans. Philips Tech. Rev. 11, 260–270 (1950).

Hamaker, H. C., Taudin Chabot, J. J. M., and Willemze, F. G. The practical application of sampling inspection plans and tables. Philips Tech. Rev. 11, 362–370 (1950).

Lawley, D. N. A further note on a problem in factor analysis. Proc. Roy. Soc. Edinburgh. Sect. A. 63, 93–94 (1950).

Corrections to a previous paper [same Proc. Sect. A. 62, 394–399 (1949); these Rev. 11, 192].

TOPOLOGY

Zykov, A. A. On some properties of linear complexes. Mat. Sbornik N.S. 24(66), 163–188 (1949). (Russian)

The author presents a theory of graphs. A graph L of order n is a set of n elements called vertices of L, certain pairs of which may be specified as edges of L. The two members of an edge are called adjacent. Say that L is a full graph if each pair of vertices is an edge, and a hollow graph if it has no edges. The sum $L+M$ and product LM of two graphs L and M are defined as follows. The vertices of each are the vertices of L and M. Each has all the edges of L and M as edges and $L+M$ has no other edges. The other edges of LM are the pairs consisting of one vertex of L and a distinct vertex of M. Addition and multiplication satisfy the commutative and associative laws and multiplication is distributive over addition. Connected (simple) graphs are graphs which are not sums (products) of two nonnull nonintersecting graphs. It is shown that any graph has a unique expression as a sum of products of nonnull nonintersecting connected simple graphs.

The author considers m-distributions of a graph, that is, partitions of the vertices among m mutually exclusive classes, so that any two adjacent vertices belong to different classes. The number of distinct m-distributions of L being denoted by $n_m(L)$, the rank $r(L)$ of L is defined as the least m such that $n_m(L)>0$. The elementary properties of $n_m(L)$ are discussed. It is shown that any graph L of rank pq, where p and q are integers, can be written in the form L_1+L_2, where $r(L_1)=p$ and $r(L_2)=q$. At this stage the following definitions are introduced. The number of full graphs of i vertices which are contained in a given graph L is denoted by $q_i(L)$ ($q_0(L)$ is taken to be 1). The fullness of L is the least i such that $q_i(L)>0$. Clearly, the fullness is not greater than $r(L)$. If the fullness is $r(L)$, the graph is said to have minimal rank. A graph is saturated if adding an edge always increases the fullness. If further there is no graph of the same order and fullness having more edges, the graph is said to be maximally saturated. Two vertices A and B of a graph L are symmetric if they are nonadjacent and each vertex adjacent to one is adjacent also to the other. The graph L is called symmetric if each pair of nonadjacent vertices of L is symmetric. It is found that symmetric graphs have minimal rank and are saturated. They are identified as the products of nonintersecting hollow graphs. It is shown that a maximally saturated graph is uniquely determined by its order and fullness (to within an isomorphism). A method of constructing such a graph of given order and fullness is explained. It is shown that if L is maximally saturated and M has the same order and fullness as L, then $q_i(M) \leqq q_i(L)$ whenever $i>1$.

W. T. Tutte (Toronto, Ont.).

Bing, R. H. Partitioning a set. Bull. Amer. Math. Soc. 55, 1101–1110 (1949).

A finite collection G of connected, mutually exclusive, open subsets of a set M is said to be an ϵ-partitioning of M

provided that the diameter of each element of G is less than $\epsilon>0$, and the sum of the elements of G is dense in M. A set can be partitioned if it can be ϵ-partitioned for each ϵ. A partitioning G of M is called a refinement of a partitioning H of M provided that each element of G is a subset of some element of H. The author proves that a set can be partitioned if and only if it has the well-known property S. It is next shown that if M can be partitioned then for each $\epsilon>0$ there exists an ϵ-partitioning of M each element of which has the property S and hence can be partitioned. This leads, for a partitionable M, to the existence of a sequence $G_1, G_2, \cdots$ such that G_i is a $(1/i)$-partitioning of M and G_{i+1} is a refinement of G_i. The central theorem is that a compact partitionable continuum can be assigned a convex metric. In consequence, the same is true of every compact locally connected continuum. The metric F for M is called almost convex if for each pair of points x, y of M and each $\epsilon>0$ there is a point z of M such that

$$|F(x, z)-\tfrac{1}{2}F(x, y)|+|F(y, z)-\tfrac{1}{2}F(x, y)|<\epsilon.$$

The author concludes by showing that each connected partitionable set has an almost convex metric. [Cf. the following review.]

W. Claytor (Washington, D. C.).

Moise, Edwin E. Grille decomposition and convexification theorems for compact metric locally connected continua. Bull. Amer. Math. Soc. 55, 1111–1121 (1949).

If G is a finite collection of mutually exclusive connected open subsets of a Peano space S such that $\bar{G}^*=S$, where G^* is the set of all elements of elements of G, then G is called a connected grating decomposition of S. If the elements of G are in addition uniformly locally connected, then G is called a grille decomposition of S. A sequence $G_1, G_2, \cdots$ of grille decompositions of S is called a complete sequence of grille decompositions of S if each term of the sequence after the first is a refinement of its predecessors and only a finite number of terms of the sequence have mesh greater than any $\epsilon>0$. The author shows that every Peano space has a complete sequence of grille decompositions. He then proceeds to prove that such a space can always be given a convex metric. This result, also established independently by Bing in the paper reviewed above, answers completely and affirmatively a question originally raised by Menger [Math. Ann. 100, 75–163 (1928), see especially pp. 81–98].

W. Claytor (Washington, D. C.).

Nishimura, Toshio. Remarks on the metrization problem. Tôhoku Math. J. (2) 1, 225–228 (1950).

The following theorems are proved. Let E be a topological space admitting a nonnegative distance function ab for every $a, b\epsilon E$, such that $ab=0$ if and only if $a=b$. If, for every $\epsilon>0$, there exists a function $\varphi(\epsilon)>0$ such that $ab<\varphi(\epsilon)$ and $cb<\varphi(\epsilon)$ imply $ac<\epsilon$, then E is metrizable. Furthermore, if for every $\epsilon>0$ and every $a\epsilon E$, there exists a function $\varphi(a, \epsilon)>0$ such that $ab<\varphi(a, \epsilon)$, $cb<\varphi(a, \epsilon)$, and $cd<\varphi(a, \epsilon)$ imply $ad<\epsilon$, then E is metrizable. *E. Hewitt.*

Wallace, A. D. Group invariant continua. Fund. Math. 36, 119–124 (1949).

The author denotes by X a compact (=bicompact) connected Hausdorff space, and by Z a group, written multiplicatively, which is a topological space (but not necessarily a topological group). By f he denotes a continuous transformation of the Cartesian product of Z and X into X. The following assumptions are made on f: (a) $f(e, x)=x$ for each x in X, where e is the unit element of Z; (b) $f(z, f(z', x))=f(zz', x)$ for each x in X and z, z' in Z. Thus, on setting $z(x)=f(z, x)$ it may be easily verified that z is a homeomorphism of X onto X, and that z^{-1} is the inverse of z as a transformation. Accordingly, the author says ("somewhat incorrectly") that Z acts as a group of homeomorphisms on X. A subset A of X is called Z-invariant provided $z(A)=A$ for each z in Z. The following results are obtained. (1) If Z is Abelian then there is a Z-invariant subcontinuum having no cutpoint. Moreover, there exists in X a Z-invariant prime chain. (2) If Z is Abelian and no proper subcontinuum of X is Z-invariant, then X has no cutpoint. (3) If Z is connected and X metric, then every endpoint and every nondegenerate prime chain is invariant.

D. W. Hall (College Park, Md.).

Begle, Edward G. A fixed point theorem. Ann. of Math. (2) 51, 544–550 (1950).

In the pattern of Lefschetz' proof of his fixed point theorem [same Ann. (2) 38, 819–822 (1937)], using homology theory only, the author proves the following generalization of a theorem of Eilenberg and Montgomery [Amer. J. Math. 68, 214–222 (1946); these Rev. 8, 51]. Let Y be a bicompact lc space which is acyclic; let f be an upper semicontinuous transformation which assigns to each point y of Y an acyclic subset $f(y)$ of Y; then, for some y, $f(y)$ contains y. The coefficient group is an arbitrary field. The author makes use of his preceding paper [same vol., 534–543 (1950); these Rev. 11, 677] which proves the Vietoris mapping theorem for bicompact spaces. *A. W. Tucker.*

Eilenberg, Samuel, and Zilber, J. A. Semi-simplicial complexes and singular homology. Ann. of Math. (2) 51, 499–513 (1950).

Cette généralisation de la notion de complexe simplicial s'applique d'une part à l'homologie et la cohomologie singulières, d'autre part à divers complexes qu'on peut attacher aux groupes discrets [voir le mémoire suivant d'Eilenberg et MacLane rapporté ci-dessous]. Le présent mémoire est consacré à un exposé général et abstrait, qui sera utilisé dans le mémoire suivant.

Un complexe semi-simplicial est, en gros, une collection de "simplexes" de toutes dimensions, dans laquelle un q-simplexe détermine ses "faces" de dimensions moins de q, mais n'est pas nécessairement déterminé par elles. Les théories du "cup-" et du "cap-product", des produits de Steenrod, sont valables dans un tel complexe; on a une notion d'"application simpliciale," et une théorie des coefficients locaux. Dans le cas où le complexe semi-simplicial est le complexe $S(X)$ formé des simplexes singuliers d'un espace topologique X, connexe par arcs, on a la notion importante de sous-complexe minimal, relativement à un point-base de X; tous les sous-complexes minimaux (il en existe) sont simplicialement isomorphes entre eux, et chacun d'eux donne, à lui seul, l'homologie et la cohomologie de l'espace X.

Une structure de complexe semi-simplicial "complet" est définie par la donnée, dans un complexe semi-simplicial, d'une loi qui, à tout q-simplexe σ, à tout entier m, et à toute application faiblement monotone α de $(0, 1, \cdots, m)$ dans $(0, 1, \cdots, q)$, associe un simplexe, noté $\sigma\alpha$, de manière que $\sigma\alpha=\sigma$ quand $m=q$ et α est l'application identique, et que $(\sigma\alpha)\beta=\sigma(\alpha\beta)$. Le complexe singulier $S(X)$ possède toujours des sous-complexes minimaux qui sont complets. Dans un complexe semi-simplicial complet K, on a la notion évidente de simplexe "dégénéré"; une cochaîne est "normalisée" si

elle s'annule sur tous les simplexes dégénérés. La cohomologie de K n'est pas changée si on travaille seulement avec les cochaînes normalisées: ce résultat permettra, dans le mémoire suivant, de généraliser la théorie de la normalisation utilisée en cohomologie des groupes [cf. Eilenberg, Bull. Amer. Math. Soc. **55**, 3–37 (1949); ces Rev. **11**, 8].
H. Cartan (Paris).

Eilenberg, Samuel, and MacLane, Saunders. Relations between homology and homotopy groups of spaces. II. Ann. of Math. (2) **51**, 514–533 (1950).

Ce mémoire fait suite à un mémoire des mêmes auteurs, paru sous le même titre [mêmes Ann. (2) **46**, 480–509 (1945); ces Rev. **7**, 137]; il utilise la technique du mémoire d'Eilenberg et Zilber rapporté ci-dessus. Soit X un espace topologique, connexe par arcs. Les auteurs avaient démontré que si $\pi_i(X)=0$ pour $i<n$ et $n<i<q$, le groupe $\pi_n(X)$ détermine entièrement, d'une manière purement algébrique, la structure homologique de X pour les dimensions moins de q: les groupes d'homologie et de cohomologie de X, pour les dimensions moins de q, sont canoniquement isomorphes à ceux d'un complexe $K(\pi_n, n)$ construit à partir du groupe $\pi_n(X)$; c'est un complexe "semi-simplicial" au sens d'Eilenberg et Zilber [cf. l'analyse ci-dessus]. Dans les mêmes hypothèses, les auteurs définissent un invariant $\mathbf{k}_n^{q+1}$ de l'espace X: c'est un élément de $H^{q+1}(\pi_n, n, \pi_q)$, le $(q+1)$-ième groupe de cohomologie du complexe $K(\pi_n, n)$ relatif au groupe de coefficients $\pi_q(X)$. Cet invariant permet de déterminer explicitement le q-ième groupe de cohomologie $H^q(X, G)$ pour n'importe quel groupe de coefficients G. Lorsque $n=1$, le groupe π_1 n'est plus abélien en général; il opère dans π_q, et dans le système de coefficients locaux G; on retrouve alors les résultats d'Eilenberg et MacLane [Trans. Amer. Math. Soc. **65**, 49–99 (1949); ces Rev. **11**, 379]. Les auteurs donnent un exposé complet de la question, indépendant des mémoires antérieurs sur le même sujet. Dans un dernier paragraphe, ils se servent du fait que $K(\pi_n, n)$ est un complexe semi-simplicial "complet" (au sens d'Eilenberg et Zilber); cela leur permet de travailler seulement avec des cochaînes "normalisées." *H. Cartan.*

Seifert, H. On the homology invariants of knots. Quart. J. Math., Oxford Ser. (2) **1**, 23–32 (1950).

Let k_i $(i=1, 2)$ be an oriented knot in 3-dimensional Euclidean space R, and V_i a closed tubular neighborhood of k_i. An oriented simple closed curve, nonbounding on the torus T_i which forms the boundary of V_i, is called a meridian if it bounds in V_i and a longitude if it bounds in $R-V_i+T_i$. Let l_i be an arbitrary knot in the interior of V_i. As a (properly oriented) 1-cycle in V_i, l_i is homologous to a certain multiple $n_i \geqq 0$ of k_i. Assume that $l_2=\phi(l_1)$ for some homeomorphism ϕ of V_1 upon V_2 which carries k_1 into k_2 and meridian and longitude of V_1 into meridian and longitude, respectively, of V_2. For this it is of course necessary that $n_1=n_2$. Theorem 2 (generalizing results of Alexander [Trans. Amer. Math. Soc. **30**, 275–306 (1938)], Burau [Abh. Math. Sem. Hamburg Univ. **9**, 125–133 (1932)], and J. H. C. Whitehead [J. London Math. Soc. **12**, 63–71 (1937)]): $\Delta_{l_1}(x):\Delta_{k_1}(x^{n_1})=\Delta_{l_2}(x):\Delta_{k_2}(x^{n_2})$. Theorem 1. If furthermore $n_1=n_2=0$, the "homology invariants" of l_1 and l_2 are the same. By homology invariants of l_i the author means the homology groups and linking invariants of the g-sheeted cyclic covering manifolds M_g of R with branch line l_i for $g=2, 3, \cdots$ and the homology group (considered as group with operator) for $g=\infty$. *R. H. Fox.*

Whitehead, J. H. C. Simple homotopy types. Amer. J. Math. **72**, 1–57 (1950).

The author continues his program of translating the theory of homotopy types into purely algebraic terms, devoting his attention in this paper to "simple homotopy type" [="nucleus" in Proc. London Math. Soc. (2) **45**, 243–327 (1939)]. Many of the results in this paper are repetitions of those in other articles on the subject by the author, but the approach is radically different and serves to unify the existing theory. Two finite connected polytopes belong to the same simple homotopy type if, geometrically, they are both deformation retracts of a third polytope in which they are both embedded, the deformation retraction taking place one cell at a time. (This notion therefore refines the idea of homotopy type.) Guided by this geometric idea, the author abstracts the induced behaviour of the "chain systems" (essentially the chain groups of the universal covering with the fundamental group operating) and introduces a notion of equivalence for such systems which is rather complicated and apparently quite deep. The first part of the paper deals with the purely algebraic properties of this abstract system, and with transformations which do not affect its geometric significance. The second part begins with showing that the algebraic equivalence is combinatorially invariant when applied to a geometric situation (the topological invariance is an open question) and then goes on to show that this algebraic equivalence is in fact adequate to characterize the geometric idea of equivalent simple homotopy type for finite connected polytopes. This algebraization, with results of Higman [Proc. London Math. Soc. (2) **46**, 231–248 (1940); these Rev. **2**, 5], leads at once to the result that simple homotopy type coincides with homotopy type if $\pi_1=1$, of order 2, 3, 4, or infinite cyclic. The closing sections handle n-type from this new viewpoint, and, further, it is shown that algebraic criteria for equivalent simple homotopy type can be stated in terms of his "homotopy systems" [Bull. Amer. Math. Soc. **55**, 453–496 (1949); these Rev. **11**, 48]. *J. Dugundji.*

GEOMETRY

Deaux, R. Sur les triangles isologiques. Mathesis **59**, 44–52 (1950).

Nehring, O. Über ähnliche, seitengebundene Dreiecke. Math.-Phys. Semesterber. **1**, 305–308 (1950).

Thébault, Victor. Sur des coniques associées à un triangle. Ann. Soc. Sci. Bruxelles. Sér. I. **63**, 74–80 (1949).

Thébault, Victor. Sur les sphères de Tücker du tétraèdre. Ann. Soc. Sci. Bruxelles. Sér. I. **62**, 67–73 (1948).

Thébault, Victor. Sur des points de Gergonne et de Nagel d'un tétraèdre. Math. Gaz. **33**, 270–272 (1949).

Thébault, Victor. Sur le tétraèdre orthocentrique. Ann. Soc. Sci. Bruxelles. Sér. I. **63**, 5–10 (1949).

Thébault, Victor. Points et droites remarquables du tétraèdre. Ann. Soc. Sci. Bruxelles. Sér. I. **64**, 39–47 (1950).

Thébault, Victor. A propos du tranchet d'Archimède. Ann. Soc. Sci. Bruxelles. Sér. I. **64**, 5–12 (1950).

Court, Nathan Altshiller. Un tétraèdre spécial "de Darboux." Mathesis 59, 13–17 (1950).

Marmion, Alphonse. Sur la géométrie du tétraèdre. C. R. Acad. Sci. Paris 230, 1924–1926 (1950).

Henmi, Denzaburô. A synthetic proof of M. B. Rao's extension of Feuerbach's theorem. Sci. Rep. Tôhoku Univ., Ser. 1. 33, 62–63 (1949).

Leite, Duarte. On arcs of a cone whose lengths have a constant quotient. Gaz. Mat., Lisboa 10, no. 41–42, 19–22 (1949). (Portuguese)

Sydler, J.-P. Construction à l'aide de la règle et de l'équerre du diamètre de courbure en un point d'une conique. Elemente der Math. 5, 49–50 (1950).

van der Woude, W. On the construction of the "pseudo-ellipse." Euclides, Groningen 25, 290–293 (1950). (Dutch)

van Beylen, E. An approximate construction for the side a_n of a regular inscribed convex polygon. Nieuw Tijdschr. Wiskunde 37, 342–344 (1950). (Dutch)

Goormaghtigh, R. Sur certaines enveloppes associées aux polygones inscriptibles. Mathesis 59, 39–44 (1950).

Gambier, Bertrand. Étude d'un cercle de grandeur constante glissant sur les arêtes d'un trièdre trirectangle fixe. Mathesis 59, 18–38 (1950).

Schuh, Fred. A theorem concerning a bundle of conic sections. Nieuw Tijdschr. Wiskunde 37, 295–297 (1950). (Dutch)

Magin, Ernst. Die Beziehungen doppelt berührender Kegelschnitte zu den Sätzen von Desargues, Pascal, Brianchon und Monge. Math.-Phys. Semesterber. 1, 288–298 (1950).

Stoll, A. Die Steinersche Hypozykloide. Elemente der Math. 5, 55–60 (1950).

Niče, V. Aperçu court de la géométrie synthétique. Bull. Soc. Math. Phys. Serbie 1, no. 3–4, 73–82 (1949). (Croatian. French summary)

Bompiani, E. Sopra una nozione di antipolarità fra rette nello spazio, occorrente in dinamica. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 8, 412–422 (1949).

Starting from dynamical investigations, Signorini [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 137–142 (1946); Ann. Mat. Pura Appl. (4) 24, 1–11 (1945); these Rev. 8, 357; 9, 162] has given a construction by means of which to each line r of space is constructed the "antipolar" line r' with respect to a given ellipsoid. The author considers this construction in a geometrical way. He determines the equations of r' when those of r are given. The lines which coincide with their antipolars are investigated. A representation of the antipolarity is given by means of line coordinates; it induces a birational correspondence on Klein's quadric in [5]. *O. Bottema* (Delft).

Kibble, W. F. Regular polytopes inscribed in other regular polytopes. Math. Student 17 (1949), 26–31 (1950).

The author remarks that 8 of the 20 vertices of the dodecahedron belong to a cube, and 4 of these to a tetrahedron. Similarly in four dimensions, 120 of the 600 vertices of the 120-cell belong to a 600-cell, 24 of these to a 24-cell, 16 of these to an 8-cell (hypercube), and 8 of these to a 16-cell [Coxeter, Regular Polytopes, Pitman, New York, 1949, pp. 149, 268–272, 305; these Rev. 10, 261]. He states incorrectly that the 5-cell (simplex) cannot be inscribed in any other regular polytope [cf. A. Urech, Polytopes réguliers de l'espace à n dimensions et leurs groupes de rotations, Zürich, 1925, p. 47]. He gives a simple proof that 480 seven-dimensional simplexes can be inscribed in the seven-dimensional hypercube, and the same number of eight-dimensional cross polytopes in the eight-dimensional hypercube [cf. J. A. Barrau, Nieuw Arch. Wiskunde (2) 7, 250–270 (1906)]. He obtains some of the analogous results in still higher spaces [cf. Coxeter, J. Math. Physics 12, 334–345 (1933); to avoid possible confusion, the reader should note that the author's A_n, B_n, C_n are the reviewer's $\alpha_n, \gamma_n, \beta_n$]. *H. S. M. Coxeter* (Toronto, Ont.).

Herrmann, Horst. Vollständige regelmässige Konfigurationen. Arch. Math. 2, 207–215 (1950).

The author considers a configuration of p points, g lines, and u planes, g_0 lines and u_0 planes through each point, u_1 planes through each line, p_1 points on each line, p_2 points and g_2 lines on each plane. He calls the configuration complete if it includes both the point of intersection and the joining plane of every intersecting pair of its lines, so that $u_0 = \frac{1}{2}g_0(g_0-1)$, $u_1 = g_0-1$, $p_2 = \frac{1}{2}g_2(g_2-1)$, $p_1 = g_2-1$. From these and the obvious relations $gp_1 = pg_0$, etc., he obtains the numerical solution $p = \kappa\binom{n}{k-1}$, $g = \kappa\binom{n}{k}$, $u = \kappa\binom{n}{k+1}$, $p_1 = k$, $g_2 = k+1$, $u_1 = n-k$, $g_0 = n-k+1$. When $\kappa = 1$, this is realized geometrically in the "binomial figure $\binom{n}{k}$," which is a 3-space section of a $(k+1)$-dimensional projection of the simplex formed by n points in $(n-1)$-space. He mentions two other important instances: $n=6$, $k=3$, $\kappa = 4/5$, and $n=10$, $k=5$, $\kappa = 2/7$. Actually, these configurations 12_6 and 60_{15} were both discovered by Stephanos [Bull. Sci. Math. (2) 3, 424–456 (1879); Math. Ann. 22, 299–367 (1883), see footnote, p. 351]. The 12_6 is the celebrated figure of three desmic tetrahedra [Hudson, Kummer's Quartic Surface, Cambridge University Press, 1905, p. 1]. The 60_{15} (not to be confused with Klein's 60_{15}) can be derived from the regular 600-cell [see Coxeter, Regular Polytopes, Pitman, New York, 1949, pp. 275, 298; these Rev. 10, 261] by taking a section of its 60 diagonal lines (joining pairs of opposite vertices), its 72 diagonal planes (each containing an "equatorial" decagon), and its 60 diagonal 3-spaces (each containing an icosidodecahedron). *H. S. M. Coxeter* (Toronto, Ont.).

Tibiletti, Cesarina. L'evoluzione della geometria secondo le idee di Klein. Period. Mat. (4) 28, 13–27 (1950).

Algebraic Geometry

***Baker, H. F. The change in our view of space and the development of the theory of algebraic loci.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 11–13, Rome, 1943.

***Hodge, W. V. D. A new set of relative birational invariants of algebraic varieties.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 141–157, Rome, 1943.

An expository paper describing the main directions of progress in algebraic geometry during the 15 years preceding 1939.

Plamitzer, Antoni. Surface d'ordre 6 ayant une courbe gauche double d'ordre 6 et de genre 3. Prace Mat.-Fiz. **47**, 67–104 (1949).

The definition from which the author embarks on a lengthy description of this F^6 is that the surface is the locus of the point of intersection of corresponding planes of three collinearly related quadric envelopes. The surface, however, is plainly the transform of a general quadric by the familiar cubic transformation of space that is generated by three correlations; and there is little the author has to say which does not follow immediately from well-known properties of this transformation, or from the resulting plane representation of F^6 by means of a linear curve-system $C^6[3^2, 1^{12}]$.
J. G. Semple (London).

Galafassi, Vittorio Emanuele. Osculanti di una curva razionale ed invarianti proiettivi di contatto. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **11(80)** (1947), 261–268 (1949).

Étant donnée une courbe plane rationnelle d'ordre n et sa première osculante en un point P simple ordinaire, le rapport de leurs courbures est $n(n-2)/(n-1)^2$, indépendant du choix de P. Plus généralement, deux osculantes d'ordres respectifs p et q ont leurs courbures dans le rapport $(p, q, 1, 0) = p/(p-1)\cdot(q-1)/q$. Si P est un point d'inflexion d'ordre i, au voisinage duquel la branche linéaire admet un développement de la forme $y = Ax^{i+1} + \cdots$, le rapport des premiers coefficients pour deux osculantes d'ordres respectifs p et q est $\prod_{k=1}^{i}(p, q, k, 0)$. L'application de ce résultat aux courbes rationnelles C^n de S_r en un point P simple ordinaire, au voisinage duquel la branche admet les développements $X_h = A_h X_1^h + \cdots$, montre que le rapport des coefficients A_{h+1} relatifs aux osculantes d'ordres respectifs p et q est $\prod_{k=1}^{k=h}(p, q, k, 0)$. Toutes ces propriétés sont très élémentaires et pourraient être obtenues également au moyen de la méthode du repère mobile. *L. Gauthier* (Nancy).

Brusotti, Luigi. Curve algebriche reali nello spazio euclideo e nello spazio iperbolico. Ann. Mat. Pura Appl. (4) **29**, 35–42 (1949).

The set Γ^* of real points of a real algebraic curve of order n and genus p in the real projective plane or 3-space has a finite number $k \leqq p+1$ of connected components. If the projective space is specialized to Euclidean or hyperbolic space by designating the points of an appropriate set as ideal points (or points at infinity), the subset Γ of Γ^* consisting of points at finite distance will contain c connected components and, in general, $c > k$. The components of Γ are classified as "segments" or "circuits" according as they arise from components of Γ^* which do or do not contain ideal points. The author's earlier results [Boll. Un. Mat. Ital. 17, 214–218 (1938)] on the structure of Γ in the case of a real algebraic plane curve are here extended to certain real, nonsingular, irreducible, twisted curves in Euclidean or hyperbolic 3-space.

The results are the same in both the Euclidean and hyperbolic cases. If π_n is the maximum genus of a twisted curve of order n ($\pi_n = \frac{1}{4}(n-2)^2$ or $\frac{1}{4}(n-1)(n-3)$ according as n is even or odd), then $c \leqq \pi_n + n$ for all twisted curves of order n. There exist curves of order n and genus π_n for which the set Γ consists of $\pi_n + n$ continua, and, when this is so, there are π_n circuits and n segments. If F is a family of twisted curves of order n and genus p, and if the series of plane sections on the general curve of F is nonspecial, then $c \leqq p+n$ for any real curve of F; there exist curves in F for which $c = p+n$, and in this case Γ consists of p circuits and n segments. The same statements are proved for the family F^* of complete intersections of surfaces of orders m_1 and m_2 for which $n = m_1 m_2$ and $p = \frac{1}{2}m_1 m_2(m_1+m_2-4)+1$. The proofs are based on the so-called "method of small variations."
H. T. Muhly (Iowa City, Iowa).

Jongmans, F. Observations complémentaires sur les séries spéciales des courbes algébriques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 128–137 (1950).

The author points out that the results in a previous paper [same Bull. Cl. Sci. (5) **35**, 1027–1041, 1113–1124 (1949); these Rev. **11**, 538] can be improved somewhat without any essential change in method.

Jongmans, F. La répartition des séries linéaires spéciales sur les courbes algébriques de genre sept. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 220–237 (1950).

A special linear series g_n^r on an algebraic curve is primitive if it is complete, free from fixed points, and not partially contained in any g_{n+i}^{r+i} ($i > 0$). The purpose of the present paper is to classify curves of genus 7 according to the primitive series which they contain, and to find a minimum order plane model for each type. As the residual of a primitive series is primitive it is sufficient to consider those of order not greater than 6. I. Hyperelliptic: $C^9(A^7)$ (i.e., curve of order 9 with 7-ple point A). One g_2^1, one g_4^2, one g_6^3, compounded with g_2^1. II. Trigonal (i.e., having g_3^1 but no g_2^1): (a) $C^6(A^3)$. One g_3^1, two residual g_6^2's, one compounded with g_3^1. (b) $C^7(A^4, 2B^2)$, the $2B$ being in the first neighborhood of A. One g_3^1, one self-residual g_6^2 compounded with g_3^1. III. Quadrigonal: (a) $C^7(A^3, 5B^2)$ such that the system $C^3(A^1, 5B^1)$ is irreducible. One g_4^1, $\infty^1 g_5^1$'s, $\infty^2 g_6^1$'s. (b) $C^6(3A^2)$. As many g_4^1's as there are proper points A, two residual g_6^1's (which coincide if $3A$ are collinear). (c) $C^8(A^4, 8B^2)$, where either A, $8B$ are associated or $8B$ lie on a conic. There is an elliptic r_2^1 with which $\infty^1 g_4^1$'s and $\infty^1 g_6^2$'s are compounded, also $\infty^2 g_6^1$'s not compounded with r_2^1. IV. Pentagonal (general case): $C^7(8A^2)$, C^7 not passing through the ninth associated point of $8A$. $\infty^1 g_5^1$'s, $\infty^2 g_6^1$'s. It is left uncertain whether there can be a curve with $\infty^2 g_5^1$'s. *P. Du Val* (Athens, Ga.).

Maroni, Arturo. Sulle curve k-gonali. Ann. Mat. Pura Appl. (4) **30**, 225–231 (1949).

An algebraic curve C of genus p is said to be k-gonal if it possesses a $g^1{}_k$ but not a $g^1{}_{k-1}$. The $g^1{}_k$ is then complete, free of fixed points, and special. The subject paper is concerned with the problems of existence and classification of the special series on such curves. If C_{2p-2} is a canonical model of C in S_{p-1} (the hyperplanes of S_{p-1} cut out the canonical series on C_{2p-2}), then each set G of $g^1{}_k$ is contained in a unique S_{k-2} of S_{p-1}. As G varies in $g^1{}_k$, the S_{k-2} generates a rational normal variety $F^m{}_{k-1}$ of dimension $k-1$ and order m. For each integer i ($=1, \cdots, k-2$), $F^m{}_{k-1}$ possesses a rational normal subvariety F_i of dimension i and minimal order m_i which is ruled by linear spaces S_{i-1}. The

orders $m_1, \cdots, m_{k-2}$ are birational invariants of the curve, and C is said to be of species $[m_1, \cdots, m_{k-2}]$. By considering the series cut out on C_{2p-2} by the hyperplanes of S_{p-1} through F_i, it is shown that on every k-gonal curve C of species $[m_1, \cdots, m_{k-2}]$ there exist $k-2$ special series, $\Sigma_1 \supset \Sigma_2 \supset \cdots \supset \Sigma_{k-2}$ of dimensions

$p-2-m_1, \cdots, p-2-(m_i+i-1), \cdots, p-2-(m_{k-2}+k-3)$.

A set G of g^1_k imposes $h=k-i-1$ conditions on the sets of Σ_i which contain it, and Σ_i contains every special series g^r_n on C on which the sets of g^1_k impose h or fewer conditions.

H. T. Muhly (Iowa City, Iowa).

Masotti Biggiogero, Giuseppina. La caratterizzazione della curva di diramazione dei piani tripli, ottenuta mediante sistemi di curve pluritangenti. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **11(80)** (1947), 151–160 (1949).

Généralisant un résultat de Pompilj sur les courbes de branchement des plans multiples réglés [Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 57–74 (1946); ces Rev. **9**, 58] l'auteur étudie les courbes de branchement des plans triples, considérées comme enveloppes d'un système à trois paramètres de courbes partout tangentes. L'auteur obtient une démonstration nouvelle du théorème de Chisini et Manara [Ann. Mat. Pura Appl. (4) **25**, 255–265 (1946); ces Rev. **9**, 463]. Cette démonstration fait intervenir une suite de dix théorèmes sur les propriétés des courbes liées de façon projectivement covariante à la courbe de branchement d'un plan triple, théorèmes qui semblent susceptibles d'autres applications.

L. Gauthier (Nancy).

Masotti Biggiogero, Giuseppina. Sulla caratterizzazione della curva di diramazione dei piani quadrupli generali. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **11(80)** (1947), 269–280 (1949).

La méthode exposée par l'auteur dans un précédent mémoire [voir l'analyse ci-dessus] pour la caractérisation des courbes de branchement des plans triples est susceptible d'être appliquée au cas des plans multiples quelconques; l'auteur expose cette extension dans le cas précis des plans quadruples (au sens: surface dont l'équation est de degré 4 par rapport à l'une des variables) et obtient le théorème suivant. Pour qu'une courbe irréductible du 12ième ordre φ_{12} ayant 24 rebroussements définissant un groupe K_{24} et 12 points doubles définissant un groupe N_{24} soit courbe de branchement d'un plan quadruple, il faut et il suffit: (a) que la série complète $|R|$ découpée par les droites soit une g^{2+s}_{12} ($s \geqq 0$); (b) que les groupes K_{24} et N_{24} soient équivalents aux groupes $2R$; (c) que la g^1_{24} définie par K_{24} et N_{24} contienne un G_{24} découpé sur φ_{12} par une conique. (Il est d'ailleurs possible que la condition (c) soit conséquence des deux autres; la question de son indépendance n'a pû être éclaircie par l'auteur.) L'équation du plan quadruple, exprimée au moyen des courbes projectivement covariantes à φ_{12} est obtenue explicitement.

L. Gauthier (Nancy).

Chisini, Oscar. Dimostrazione delle condizioni caratteristiche perchè una curva sia di diramazione di un piano quadruplo. Ann. Mat. Pura Appl. (4) **29**, 15–23 (1949).

La courbe de diramation d'un plan quadruple, image d'une surface F d'ordre $4+n$ dotée d'un point multiple O d'ordre n, est la projection à partir de O de la section de F et de la polaire de O. Une telle courbe possède un ordre $m=6n+12$ avec $4(n+1)(n+3)$ points doubles et $m(n+2)$ cuspides; les groupes des points cuspidaux K des points doubles N considérés sur chaque rameau vérifient les relation fonctionnelles: $K \equiv (n+2)R$, $N+4T \equiv 2(n+1)R$, $R+T \equiv Z$, R étant un groupe de points en ligne droite, T un groupe de points non virtuels, et Z un groupe de points tous différents des T. Partant d'une courbe qui possède ces caractères, et menant la démonstration dans le cas plus simple $n=5$, Chisini montre que l'on peut reconstruire la surface dont le plan quadruple doté de cette courbe de diramation est l'image. Pour ce faire il généralise certains résultats de Masotti [voir l'analyse ci-dessus]. Il montre que les relations fonctionnelles permettent de définir certaines courbes dont l'introduction permet d'écrire diverses équations de la diramante et de construire une courbe dont elle soit projection appartenant à des F^n dotées d'un point quintuple au centre de projection; la courbe intersection se montre alors section de l'une d'elles par sa polaire. Un passage à une forme décomposée de la diramante montre que toutes les surfaces que l'on pourrait ainsi construire sont birationnellement identiques.

B. d'Orgeval.

Lense, Josef. Bestimmung einer ebenen birationalen quadratischen Verwandtschaft durch sieben Paare entsprechender Punkte. Math. Z. **52**, 605–610 (1950).

Any plane quadratic (birational) transformation, envisaged as a two-fold variety of pairs of corresponding points, is the base of a unique linear pencil of correlations, envisaged as three-fold varieties of pairs of conjugate points. This theorem, of which a rigorous proof is here given, leads at once to the result that seven point-pairs of general position determine a unique quadratic transformation in which they correspond. In other words, if $V_4^6[8]$ is the Segre variety mapping the pairs of points of two planes, or the ordered point-pairs of one plane, the surfaces on V_4^6 which map all quadratic transformations are its [6]-sections, apart from those which are degenerate or of dimension 3.

J. G. Semple (London).

Dantoni, Giovanni. Sulla possibilità di decomporre una corrispondenza cremoniana fra spazi ad $r \geqq 3$ dimensioni, nel prodotto di corrispondenze cremoniane di dato ordine. Ann. Mat. Pura Appl. (4) **29**, 243–246 (1949).

The author proves very simply that in space of r ($\geqq 3$) dimensions there exists, for any given value of n, a Cremona transformation which cannot be expressed as the product of Cremona transformations of order less than or equal to n.

J. A. Todd (Cambridge, England).

Zappa Casadio, Giuseppina. Determinazione delle trasformazioni cremoniane fra due spazi, aventi infinite direzioni inflessionali incidenti ad una curva data. Giorn. Mat. Battaglini (4) **3(79)**, 63–65 (1950).

In any Cremona transformation T between spaces S_3 and S_3', a direction t at a point P of S_3 is said to be inflexional if any curve which has t as inflexional tangent at P transforms to a curve of S_3' which has the corresponding direction t' as inflexional tangent at the point P' corresponding to P. Suppose now that there exists in S_3 a curve φ such that the direction of any line intersecting φ is inflexional at every point of the line. It is proved then that φ is either a conic, in which case T is a quadratic transformation of the first kind, or a line, in which case T is generated by a homaloidal net of ruled surfaces of order n with φ as $(n-1)$-fold line.

J. G. Semple (London).

Godeaux, Lucien. Une représentation des transformations birationnelles du plan et de l'espace. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **24**, no. 2, 31 pp. (1949).

Ce mémoire est un exposé d'ensemble de la théorie des transformations crémoniennes, dressé au moyen d'une méthode mise au point simultanément et indépendamment par Godeaux [Bull. Soc. Roy. Sci. Liége **11**, 428–432 (1942); ces Rev. **7**, 74] et par Fano [Comment. Math. Helv. **14**, 193–201 (1942); ces Rev. **3**, 304] et étudiée sur des exemples par Calvo [Bull. Soc. Roy. Sci. Liége **11**, 166–170, 532–547 (1942); **12**, 407–415, 62–73 (1944); Acad. Roy. Belgique. Bull. Cl. Sci. (5) **29**, 657–665 (1943); ces Rev. **7**, 74, 172]. Dans une première partie, l'auteur étudie les transformations birationnelles T du plan; les couples de T sont représentés par les points d'une surface F^{2n+2} de S_{n+4} rationnelle normale à sections de genre $n-1$ sur laquelle les droites (ou leurs transformées) ont pour image un réseau homaloïdal de C^{n+1} (ou C'^{n+1}) rationnelles normales. Un point fondamental d'ordre s à tangentes variables a pour image une courbe G rationnelle normale d'ordre s de degré virtuel -1, et l'expression d'une courbe C' (ou C) au moyen de combinaisons linéaires fonctionnelles des G (ou G') permet de retrouver très rapidement tous les résultats classiques sur les points fondamentaux. Les points fondamentaux à tangentes fixes entraînent la décomposition de la courbe image sur F: cette surface peut alors présenter un point multiple.

Dans la seconde partie du mémoire l'auteur applique la même méthode à l'étude des transformations birationnelles de l'espace. Il retrouve ainsi la quasi-totalité des résultats classiques de Montesano [Soc. R. Napoli. Atti R. Accad. Sci. Fis. Mat. (2) **17**, no. 8 (1926)] en leur donnant une signification fonctionnelle très suggestive. Le mémoire se termine par la représentation des transformations birationnelles involutives au moyen d'homologies harmoniques hyperspatiales. *L. Gauthier* (Nancy).

Godeaux, Lucien. Sur le calcul des invariants d'une surface multiple ayant un nombre fini de points de diramation. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 170–179 (1950).

Soit Φ la surface image d'une involution cyclique d'ordre premier p ($p \neq 2$) appartenant à une surface algébrique F, et n'ayant qu'un nombre fini de points unis. En utilisant une méthode due à Severi [Ist. Lombardo Sci. Lett. Rend. (2) **36**, 495–511 (1903)], l'auteur détermine le genre linéaire $\pi^{(1)}$ et l'invariant de Zeuthen-Segre J de Φ en fonction de ceux $p^{(1)}$ et I de F. En appliquant ensuite la relation de Noether, on en tire le genre arithmétique π_a de Φ au moyen de celui p_a de F sous la forme: $12(p_a+1)=12p(\pi_a+1)+\sum\delta_i$, où les δ_i sont les contributions apportées par les divers points de ramification. L'auteur donne une expression précise des nombres δ_i et les étudie dans divers cas particuliers poussés jusqu'aux exemples numériques. *L. Gauthier* (Nancy).

Godeaux, Lucien. Involutions irrégulières appartenant à la surface des couples de points d'une courbe algébrique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 102–112 (1950).

Godeaux, Lucien. Sur la variété des cordes d'une courbe rationnelle normale. Bull. Soc. Roy. Sci. Liège **19**, 9–14 (1950).

Turri, Tullio. Appunti a recensione di nota sulle trasformazioni antibirazionali involutorie del piano. Rend. Sem. Fac. Sci. Univ. Cagliari **17** (1947), 95–97 (1948).

Verf. hatte früher [Rend. Sem. Fac. Sci. Univ. Cagliari **15** (1945), 189–192 (1947); diese Rev. **9**, 199] den Satz ausgesprochen, dass jede involutorische antibirationale Transformation der Ebene dem Produkt einer ebenen reellen quadratischen Transformation und dem Konjugium birational äquivalent ist. Ref. hatte Zweifel am diesem Satz und gab in dem Referat [loc. cit.] einige Gegenbeispiele. Verf. zeigt nun in vorliegender Abhandlung, dass diese die Aussage nicht treffen. In einer weiteren Arbeit [siehe nachstehendes Referat] gibt er neben anderem einen neuen Beweis seines Satzes. *O.-H. Keller* (Münster).

Turri, Tullio. Sulle coppie di trasformazioni piane birazionali involutorie permutabili. Rend. Sem. Fac. Sci. Univ. Cagliari **18** (1948), 23–28 (1949).

Wenn von zwei vertauschbaren ebenen involutorischen Cremona-Transformationen keine einer projektiven Involution birational äquivalent ist, so sind sie zwei anderen Cremona-Transformationen birational äquivalent, deren Produkt eine Projektivität ist. Verf. beweist damit folgenden Satz, den er früher behandelt hatte [siehe vorstehendes Referat]. Jede antibirationale involutorische Transformation der Ebene ist birational äquivalent dem Produkt einer reellen Involution zweiter Ordnung mit der Transformation des Konjugiums. *O.-H. Keller* (Münster).

Turri, Tullio. Una proprietà della curva di punti uniti in una trasformazione birazionale del piano. Rend. Sem. Fac. Sci. Univ. Cagliari **18** (1948), 29–31 (1949).

Ein vielfacher Punkt der Kurve der Fixpunkte einer ebenen Cremona-Transformation ist ein Fundamentalpunkt derselben. Stimmen die Fundamentalpunkte einer ebenen Cremona-Transformation mit denen der inversen Transformation überein und gibt es eine Kurve von Fixpunkten, die keine Gerade ist, so ist die Transformation involutorisch. *O.-H. Keller* (Münster).

Franchetta, Alfredo. Sulle curve riducibili appartenenti ad una superficie algebrica. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 378–398 (1949).

On dit qu'une courbe tracée sur une surface algébrique F est virtuellement connexe si deux parties complémentaires quelconques (même coïncidentes) ont un nombre positif d'intersections virtuelles. Une courbe virtuellement connexe est de genre non négatif. Dans toute transformation birationnelle, la transformée d'une courbe virtuellement connexe est encore virtuellement connexe. L'auteur établit une condition suffisante de virtuelle connexité qui lui permet de montrer qu'une courbe C_0, limite d'une courbe C irréductible de degré positif, est virtuellement connexe. La limite d'une courbe irréductible de degré nul, bien que topologiquement connexe, peut ne pas être virtuellement connexe: deux parties complémentaires ont un nombre d'intersections non négatif, et qui s'annule dans le seul cas où la composante a un point multiple commun avec la courbe entière. Ce résultat complète le principe de décomposition de Noether-Enriques, que l'auteur applique à l'étude du rôle joué par les courbes décomposées dans le calcul de l'invariant de Zeuthen-Segre. Dans la dernière partie du mémoire, l'auteur démontre que si $|L|$ est un système linéaire sans point base sur les courbes fondamentales et si C est une courbe virtuellement connexe formée au moyen de celles-ci, quand une courbe de $|L|$ contient un point de C, elle contient C tout

entière. Ce théorème a été déja utilisé (sans démonstration) par Enriques [Revista Acad. Ci. Madrid **40**, 149–159 (1946); ces Rev. **9**, 373]. L'auteur examine, pour terminer, les cas où $|L|$ a des points bases sur ses courbes fondamentales.

L. Gauthier (Nancy).

Franchetta, Alfredo. Sul sistema aggiunto ad una curva riducibile. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 685–687 (1949).

Soit C une courbe de genre virtuel p sur une surface F de genre arithmétique p_a. Le système adjoint $|C'|$ a pour dimension $p_a+p-1+i$, $i\geqq 0$. Il est régulier lorsque $i=0$. Des conditions suffisantes de régularité ont été données par Picard [1905], puis étendues récemment par Severi [Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 346–352 (1947); ces Rev. **9**, 461]. L'auteur démontre d'abord deux lemmes. Si $|C'|$ est régulier et si L désigne une courbe irréductible de F ayant un nombre positif d'intersections virtuelles avec C, le système adjoint à $C+L$ est régulier. Soit une composante $\bar{C}$ d'une courbe C virtuellement connexe, c'est à dire, telle que deux parties complémentaires quelconques, éventuellement coïncidentes, ont un nombre positif d'intersections virtuelles. Si $|\overline{C'}|$ est régulier, il en est de même de $|C'|$. En associant ces deux lemmes à la condition de Severi, l'auteur obtient la condition plus large. Pour que $|C'|$ soit régulier, il suffit que C soit virtuellement connexe et qu'une de ses composantes irréductible soit courbe totale d'un système algébrique à un paramètre, différent d'un faisceau irrationnel. *L. Gauthier* (Nancy).

Vaccaro, Giuseppe. Ricerche sugli spazi lineari di una ipersuperficie algebrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 31–34 (1950).

L'auteur démontre synthétiquement au moyen du système polaire, puis vérifie analytiquement les deux théorèmes suivants. La condition nécessaire et suffisante pour qu'une V_{r-1}^n de S_r ait lelong d'un S_k, qu'elle contient simplement, un S_{r-k} $(r-k\geqq k)$ tangent fixe, est qu'il existe dans S_k une $V_{k-h}^{(n-1)h}$ de points doubles de V_{r-1}^n. La condition nécessaire et suffisante pour qu'une V_{r-1}^n de S_r ait lelong d'un S_k, qu'elle contient avec la multiplicité s, un cône de tangentes fixes de dimension $r-h$ est qu'il existe dans S_k und $V_{k-h}^{(n-s)h}$ de points ayant sur $V^n{}_{r-1}$ une multiplicité au moins égale à $s+1$. Ces deux démonstrations sont très élémentaires.

L. Gauthier (Nancy).

Derwidué, L. Le problème général de la réduction des singularités d'une variété algébrique. Mém. Soc. Roy. Sci. Liége (4) **9**, no. 2, 139 pp. (1949).

This is an elaboration of earlier work of the author [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 399–412, 432–444 (1948); these Rev. **10**, 322]. The general method is the same, but the details of the process are emphasized. The basic reduction applies to a V_k in S_r, with $r\geqq 4k$, but an additional chapter treats the case of a hypersurface in S_4 and also of a linear system of such hypersurfaces. In spite of the emphasis on detail there are parts of the argument (particularly when use is made of properties of neighboring points or of "general" elements) which are not entirely convincing. *R. J. Walker* (Ithaca, N. Y.).

Derwidué, L. Méthode simplifiée de réduction des singularités d'une variété algébrique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 880–885 (1949).

A much simpler variation of the above proof, using general birational instead of Cremona transformations. The same criticism applies. *R. J. Walker* (Ithaca, N. Y.).

Derwidué, L. Réduction des singularités d'une surface algébrique. Bull. Soc. Roy. Sci. Liége **18**, 415–420 (1949).

Derwidué, L. Réduction des singularités d'une variété algébrique à trois dimensions. Bull. Soc. Roy. Sci. Liége **18**, 421–430 (1949).

Special cases of the above proof, with appropriate simplifications. *R. J. Walker* (Ithaca, N. Y.).

Derwidué, L. Sur les courbes exceptionnelles. Bull. Soc. Roy. Sci. Liége **18**, 431–444 (1949).

Application of the above methods to the elimination of the exceptional curves of the first kind. *R. J. Walker.*

Derwidué, L. Sur la décomposition des transformations birationnelles. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 238–250 (1950).

Soit (V_k) une classe de variétés algébriques (au sens de l'équivalence birationnelle) et soit $G(V_k)$ le groupe des permutations de tous les modèles projectifs de (V_k). L'auteur se propose de déterminer une factorisation de G. On a successivement: $G=T_2^{-1}G'T_1$, $G'=T_2'^{-1}G''T_1'$ où G' est le groupe des permutations opérant sur les modèles projectifs dépourvus de singularités, et G'' le sousgroupe de G' constitué par des correspondances strictement biunivoques. Les transformations T dites élémentaires de première espèce permutent un modèle donné arbitraire avec un modèle sans singularité. L'existence de T et sa structure sont indiquées rapidement comme application d'une méthode élaborée dans un mémoire antérieur [voir l'analyse ci-dessus du second mémoire de l'auteur] et j'avoue n'avoir pas très bien vu dans quelle mesure les affirmations du texte sont effectivement démonstratives (existence de systèmes linéaires totalement simples de base donnée; décroissance effective des nombres d'intersection). Les transformations T' dites élémentaires de seconde espèce sont obtenues par une généralisation rapidement esquissée des T de première espèce.

L. Gauthier (Nancy).

Giambelli, Giovanni. Problemi di Clebschiani di spazi. Ann. Mat. Pura Appl. (4) **29**, 163–170 (1949).

The author outlines some properties of Clebschians which, roughly speaking, may be regarded as intersections of connexes; he is concerned in particular with the Clebschians defined by the vanishing of all the minors of given order in a matrix the elements of which are polynomials in several sets of variables, and obtains a formula for certain characters of the construct which generalise the notion of the order of an ordinary algebraic variety. Various extensions and applications are indicated briefly. *J. A. Todd.*

Differential Geometry

Gürsan, Feyyaz. Les évolutoïdes. Bull. Tech. Univ. Istanbul **2**, 79–85 (1949). (French. Turkish summary)

Let the point A trace a rectifiable curve (A) with arc length σ in plane Euclidean geometry. Let $\mathbf{u}_0$ be a fixed unit vector in the plane, and with the help of two differentiable functions $\theta=\theta(\sigma)$, $\lambda=\lambda(\sigma)$, define a unit vector $\mathbf{u}$ and a point B by the relations: angle $(\mathbf{u}_0, \mathbf{u})=\theta$, $\overline{AB}=-\lambda\mathbf{u}$. The point B then traces a curve (B) called the evolutoid of (A) with respect to the functions θ and λ. Formulas for the arc length s of (B) and the angle ϕ between $\mathbf{u}$ and the tangent

to (B) at B are developed. The author then studies evolutes, tractrices, and pursuit curves by making suitable assumptions on ϕ, θ, λ, or s. For instance, for pursuit curves ($\phi=0$ and $s=k\sigma$, k constant) he shows that for arbitrary (A) λ vanishes for a value of $\sigma\leqq\lambda(0)/(k-1)$ if $k>1$. Generalized pursuit curves ($s=k\sigma$, $\phi>0$) are also mentioned and studied quite simply in the case where (A) is a straight line.

A. Schwartz (New York, N. Y.).

Kruppa, Erwin. Das Analogon zu einem Satz von Cesàro über Bertrand-Kurven im Bereich der Strahlflächen. Monatsh. Math. **54**, 45–54 (1950).

Let S be a point on the striction curve of a ruled surface in Euclidean three-space. Let $C(S)$ be the right-handed orthogonal coordinate system with S as origin, with the ruling through S as first axis and with the normal to the surface at S as second axis. Let $a(S)$ denote a straight line associated with S whose position with respect to $C(S)$ is fixed as S moves along the striction curve. The author answers the following questions. (1) For which ruled surfaces can $a(S)$ be chosen such that the trajectories of its points intersect $a(S)$ orthogonally as S moves along the striction curve? (2) What is the locus with respect to $C(S)$ of all straight lines $a(S)$ with the above property? The author distinguishes 11 types of such ruled surfaces.

W. van der Kulk (Providence, R. I.).

Wunderlich, W. Raumkurven, die pseudogeodätische Linien zweier Kegel sind. Monatsh. Math. **54**, 55–70 (1950).

By a pseudogeodesic curve on a surface is meant a curve whose osculating plane is inclined at a constant angle γ to the tangent plane of the surface. When $\gamma=\frac{1}{2}\pi$ the curve is an ordinary geodesic, and when $\gamma=0$ it is an asymptotic line. Let a curve l cut the generators of a cone with vertex O at a constant angle. Corresponding to it in the polarity determined by a sphere with O as centre is a curve k whose osculating plane cuts the cone at a constant angle. This duality between loxodromic curves and pseudogeodesic curves on a cone constitutes the main tool of the investigation. The author shows that there exist space curves which are pseudogeodesics of two cones with given vertices O_1 and O_2. Space curves which are at the same time pseudogeodesics of a cone and of a cylinder can be treated as a limiting case of two cones. *E. T. Davies* (Southampton).

Tietze, Heinrich. Die Relation zwischen den drei quadratischen Fundamentalformen einer Fläche. Math. Z. **52**, 590–592 (1950).

If I, II, III are the three quadratic forms associated with a surface in Euclidean three-space then $\mathrm{III}-2H\mathrm{II}+K\mathrm{I}=0$, where K is the curvature and H the mean curvature of the surface. The author proves this theorem by means of the well known fact that every matrix satisfies its characteristic equation. *W. van der Kulk* (Providence, R. I.).

Fabricius-Bjerre, Fr. Über projektive Böschungslinien auf Flächen 2. Ordnung. Danske Vid. Selsk. Mat.-Fys. Medd. **25**, no. 17, 21 pp. (1950).

This paper is concerned with space curves, denoted by σ, whose tangents cut a given conic a. The author confines himself to curves σ which lie upon a nondegenerate surface Φ of the second order. If the tangent plane π at any point P of Φ cuts the conic a in two points, then two curves σ pass through P. If π does not cut a at all, then no curves σ pass through P. If π touches a, then P must lie on the curve of section of Φ with the polar cone Γ of a with respect to Φ. This curve of section is a curve C^4 of the 4th order which divides Φ into two regions, in one of which curves σ exist, and in the other no σ exist. The author next considers an arbitrary point Q on a and examines if there are σ on Φ whose tangents pass through Q. The polar plane of Q with respect to Φ is a tangent plane to the polar cone Γ and cuts Φ in a conic q. If q is real then one σ passes through each of its points. The conic q has double contact with C^4, and the points of contact for different points Q on a lie upon the generators in which the plane of q cuts the polar cone Γ of a. The tangent to q at any point P on it and the straight line QP are conjugate lines with respect to Φ. Results are deduced by projection of the figure consisting of Φ, a, C^4, q, and σ, from a point O outside Φ onto any plane not passing through O. Finally the curves σ on convex surfaces of the second order are determined. *E. T. Davies.*

Blaschke, Wilhelm. Sulla geometria differenziale delle superficie S_2 nello spazio euclideo S_4. Ann. Mat. Pura Appl. (4) **28**, 205–209 (1949).

The tangent planes to a surface S_2 in Euclidean E_4 meet the elliptic P_3 at infinity in a rectilinear congruence. The study of S_2 through this associated congruence leads to two Levi-Civita parallelisms, two "total" curvatures, and two analogues of the Gauss-Bonnet formula. Noteworthy is the use of quaternions and the exterior calculus.

J. L. Vanderslice (College Park, Md.).

Inzinger, Rudolf. Berührungsinvarianten von Elementvereinen. Ann. Mat. Pura Appl. (4) **28**, 149–152 (1949).

This is a study of the geometry of the second order surface elements at a given first order surface element E_1 (of a two parameter family M) under contact transformations leaving the given first order element fixed. Engel coordinates, homogeneous and quadratically supernumerary, of these second order elements furnish a projective classification and a linear representation of the group G_{10} of the above contact transformations. They also have special meaning as coordinates of the indicatrix of M in E, both as point and line locus. Thus the study is brought within the frame of projective invariant theory. *J. L. Vanderslice.*

Pastori, Maria. Significato degli invarianti intrinseci di un tensore doppio simmetrico e curvature di una varietà. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 11(80) (1947), 7–17 (1949).

Intrinsic invariants of a given tensor in a Riemannian space are those formed by combining it with the fundamental metric tensor (first kind) or with the skew-symmetric Ricci tensor (second kind). The intrinsic invariants of a double symmetric tensor in Riemannian spaces of two and three dimensions are interpreted in terms of the characteristic roots associated with the double tensor. Applications are made to the second fundamental quadratic form of a surface and to the contracted Riemann-Christoffel tensor.

A. J. McConnell (Dublin).

Lichnerowicz, André. Dérivation covariante et nombres de Betti. C. R. Acad. Sci. Paris **230**, 1248–1250 (1950).

Two theorems are stated for a compact orientable Riemannian n-space with positive definite metric $ds^2=g_{ij}dx^idx^j$. (1) If there is a second order symmetric tensor T_{ij}, not proportional to g_{ij}, whose covariant derivative is zero, then

the space is locally reducible and its Betti numbers satisfy the inequalities $b_r \geqq p_r$ $(r=1, 2, \cdots, n-1)$, where p_r is the number of characteristic values of T_{ij} of multiplicity r. (2) If the space is recurrent (i.e., its curvature tensor $R_{ij,hk}$ satisfies $\nabla_l R_{ij,hk} = K_l R_{ij,hk}$ for some vector K_l), is locally irreducible, and is such that the curvature tensor admits a simple characteristic value in the tangent bivector space, then n is even and $b_{2k} \geqq 1$, $k=1, 2, \cdots, \frac{1}{2}n-1$. Detailed proofs of these theorems are not given but will appear elsewhere. *A. G. Walker* (Sheffield).

Segre, Beniamino. Geometria non euclidea ed ottica geometrica. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 7, 16–19 (1949).
Segre, Beniamino. Geometria non euclidea ed ottica geometrica. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 7, 20–26 (1949).

The following theorem is proved. Riemannian spaces of constant curvature are the only Finsler spaces that can be represented in the real domain on a Euclidean space in such a way that their hyperspheres correspond to those of the representative space, and every such representation is necessarily conformal. An application can be made to geometrical optics. If a Euclidean space S_n is filled with a crystalline medium, we associate with it a "Fermat space" V_n, whose line-element is $ds = \nu dl$, where dl is the line-element of S_n and ν is the index of refraction of the medium. The space V_n is a Finsler space, in which the light rays are represented by geodesics, and V_n becomes a Riemannian space when the medium is isotropic. Using the above theorem and results of previous papers [Segre, same Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 393–397, 547–550, 661–667 (1949); 7, 12–15 (1949); these Rev. 11, 541], the following results are established. If an optical medium in S_n is isotropic and such that light is propagated in it along curves having constant curvature, then the associated Fermat space is Riemannian and of constant curvature. Also there are only five types of optical media having this property, (i) homogeneous media, (ii) isotropic media whose ν is inversely proportional to the distance from a fixed plane, (iii) isotropic media whose ν is inversely proportional to the square of the distance from a fixed point, (iv and v) isotropic media whose ν is given by $\nu = a/(x_1^2+x_2^2+\cdots+x_n^2 \pm b)$, a, b constant, where $(x_1, \cdots, x_n)$ are the Cartesian coordinates of S_n.
A. J. McConnell (Dublin).

Please see the note to p. 241, Tsuji

Iwamoto, Hideyuki. Über eine geometrische Theorie der mehrfachen Integrale. Jap. J. Math. 19, 479–512 (1948).

L'auteur donne une méthode permettant d'attacher intrinsèquement une géométrie à connexion euclidienne d'éléments de contact à l'intégrale multiple d'ordre K

$$I = \int L\left(x^i, \cdots, \frac{\partial^M x_i}{\partial u^{\lambda_1} \cdots \partial u^{\lambda_M}}\right) du^1 \cdots du^K, \quad i=1, 2, \cdots, N,$$

calculée sur une sous-variété analytique à K dimensions d'une variété analytique V_N à N dimensions. Le problème se présente comme la généralisation naturelle de celui qui conduit aux géométries de Finsler ($K=1$, $M=1$), de Cartan ($K=N-1$, $M=1$), de Kawaguchi ($K=1$, M quelconque). Après avoir étudié les espaces fibrés des éléments plans d'ordre $1, 2, \cdots, M$ tangents à une V_N, l'auteur étudie en détail le cas ($M=1$, K quelconque) et détermine complètement un tenseur métrique et une connexion euclidienne d'éléments de contact intrinsèquement attachée à l'intégrale I. L'existence et l'unicité du tenseur métrique sont démontrées sous des hypothèses assez larges, par un procédé transcendant et par recours à des résultats classiques de la théorie des groupes. Les raisonnements précédents sont ensuite utilisés et étendus au cas M quelconque et une connexion euclidienne répondant à la question est effectivement déterminée. *A. Lichnerowicz* (Paris).

Pallu de La Barrière, Robert. Sur les formules de transformation des intégrales multiples. Norske Vid. Selsk. Forh., Trondheim 21, no. 7, 28–31 (1948).

L'auteur définit, dans l'espace euclidien à n dimensions, des formes "symboliques" extérieures Ω dont les coefficients sont des distributions au sens de L. Schwartz. Il montre comment ces formes peuvent être introduites d'une manière naturelle dans les différentes expressions de la formule générale de Stokes. *A. Lichnerowicz* (Paris).

Pallu de La Barrière, Robert. Sur une généralisation des formes différentielles extérieures. Norske Vid. Selsk. Forh., Trondheim 21, no. 9, 35–37 (1948).

Cette note est consacrée à la définition des opérations essentielles sur les formes Ω ayant pour coefficients des distributions de Schwartz [voir la note précédente]: différentiation, multiplication d'une forme Ω par une forme dont les coefficients sont des fonctions indéfiniment dérivables. Le produit de deux formes Ω s'effectue selon les règles du produit de composition. Le théorème de Poincaré sur les formes qui sont la différentielle d'une forme s'étend aux formes Ω sans difficultés particulières. *A. Lichnerowicz.*

Ôtsuki, Tominosuke. On the spaces with normal conformal connexions and some imbedding problem of Riemannian spaces. I. Tôhoku Math. J. (2) 1, 194–224 (1950).

The paper studies spaces with normal conformal connection whose group of holonomy fixes one or two hyperspheres, problems initiated by Sasaki and Yano. If the group of holonomy fixes one hypersphere, the space is conformal with an Einstein space. Theorems are also established on the possibility of imbedding Riemann spaces as hypersurfaces of an Einstein space. *S. Chern* (Chicago, Ill.).

Bompiani, Enrico. Interpretazione proiettiva degli spazi a connessione affine. Ann. Mat. Pura Appl. (4) 28, 69–87 (1949).

In a previous paper [Ann. Mat. Pura Appl. (4) 24, 257–282 (1945); these Rev. 9, 158] the author set about geometrizing the theory of an affinely connected space V_n. He continues in this direction but by a somewhat different path which leads to a union of ideas from projective, algebraic, and differential geometry. The V_n is considered as immersed in an S_N (projective space of N dimensions) where $N \geqq \frac{1}{2}n(n+3)$, the maximal dimensionality of the second order osculating spaces $S(2)$ of V_n. To assign paths to V_n is equivalent to assigning in a hyperplane of each $S(2)$ an algebraic variety W constructed by means of a projectivity between the points of an S_{n-1} and a Veronese variety. In turn W determines a family (S_p) of flat p-spaces, any one of which can be used as "support" for the paths regarded as a pluriaxial curve system. To fix the affine connection among those having these paths is to fix one S_p of the family. The osculating plane to the path at the point of V_n in question meets this S_p in a line, the affine normal. This normal is used to define geometrically affine arc length along the path.
J. L. Vanderslice (College Park, Md.).

Bompiani, E. **Geometria delle equazioni e dei sistemi di equazioni differenziali ordinarie.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 409–441, Rome, 1943.

This is an expository paper giving a brief résumé of a large part of the modern work on the geometry of differential equations up to 1939. It could well form a chapter in a mathematical encyclopedia except that in the final section the author makes suggestions as to fruitful directions for further investigations. It runs the gamut from the linear homogeneous ordinary equation (projective geometry of a curve) to systems of nonlinear partial differential equations such as those of the Douglas geometry of k-spreads and its later generalizations by Kawaguchi and his school. There is an extensive bibliography.

J. L. Vanderslice (College Park, Md.).

NUMERICAL AND GRAPHICAL METHODS

Marcum, J. I. Tables of Hermite polynomials and the derivatives of the error function. The RAND Corporation, Santa Monica, Calif., Report **P-90**, unpaged (1948).

The tables give the derivatives of the error function, namely

$$\frac{d^n}{dx^n}\left\{\frac{1}{(2\pi)^{\frac{1}{2}}}e^{-\frac{1}{2}x^2}\right\},$$

and values of the Hermite polynomials [note the missing − sign as given in the introduction]

$$(-)^n H_n(x) = e^{\frac{1}{2}x^2}\frac{d^n}{dx^n}e^{-\frac{1}{2}x^2},$$

for $n=1(1)10$, $x=0(.01)12$ with 6 significant figures throughout. They are more extensive than any previously available table known to the reviewer, although the table in J. W. Glover's "Tables of Applied Mathematics in Finance, Insurance and Statistics" [Wahr, Ann Arbor, Mich., 1923] comes nearer to the author's requirements than any he mentions, giving 5 decimals for $n=2(1)8$, $x=0(.01)4.99$.

J. C. P. Miller (London).

Foks, L. [Fox], Haski, H. D. [Huskey], and Vilkinson, Dž. H. [Wilkinson]. Notes on the solution of simultaneous linear algebraic equations. Uspehi Matem. Nauk (N.S.) **5**, no. 3(37), 60–86 (1950). (Russian)

Translation of a paper in the Quart. J. Mech. Appl. Math. **1**, 149–173 (1948); these Rev. **10**, 152.

Bückner, Hans. Über ein unbeschränkt anwendbares Iterationsverfahren für Systeme linearer Gleichungen. Arch. Math. **2**, 172–177 (1950).

To solve the system of k equations in k unknowns $Ax-b=0$ where $\|A\|\neq 0$ one may use the iteration process $x^{(n+1)}=x^{(n)}+c_n(Ax^{(n)}-b)$, where the c_n are suitable constants periodic in n with period p. Using the latent roots of the matrix A, the author shows how $c_1, c_2, \cdots, c_p$ can be chosen so that the iteration process always converges. In particular, if the latent roots are all real we may take $p=2$, $c_1=1-a$, $c_2=1+a$, with a sufficiently small. The case of homogeneous equations where $\|A\|=0$ is also treated.

W. E. Milne (Los Angeles, Calif.).

Gatto, Franco. Sulla risoluzione numerica dei sistemi di equazioni lineari. Ricerca Sci. **19**, 1385–1388 (1949).

This paper recommends certain convenient tabulation set-ups for the numerical work of solving linear equations by iteration. *W. E. Milne* (Los Angeles, Calif.).

Volta, Ezio. Un nuovo metodo per la risoluzione rapida di sistemi di equazioni lineari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7** (1949), 203–207 (1950).

If A is the augmented matrix for a set of n equations in n unknowns, the first step of the proposed solution is to find a square matrix B_1 such that $B_1A=A_1$ where the first column of A_1 is $1, 0, \cdots, 0$. Then find B_2 so that $B_2A_1=A_2$ where the first two columns of A_2 are respectively $1, 0, \cdots, 0$ and $0, 1, \cdots, 0$. Continuation of this process evidently leads to the solution. *W. E. Milne* (Los Angeles, Calif.).

Ricci, Lelia. Confronto fra i metodi di Banachiewicz, Roma e Volta per la risoluzione dei sistemi di equazioni algebriche lineari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7**, 72–76 (1949).

The above methods are first compared with respect to the number of additions, multiplications, etc., required by each. It is then pointed out that this comparison does not necessarily furnish a correct criterion for the most convenient method of solution. (For instance, in obtaining a sum of products on the calculating machine the number of additions should not be counted, as the addition is automatic.) The methods are compared with respect to the convenience of the scheme of calculation, and also with respect to more subjective considerations such as nervous tension, weariness, and distractions on the part of the operator. The note concludes with a comparison of time actually used in the solution of specific numerical examples. In the main the author's results appear to favor Banachiewicz.

W. E. Milne (Los Angeles, Calif.).

Bodewig, E., und Zurmühl, R. Zu R. Zurmühl: Zur numerischen Auflösung linearer Gleichungssysteme nach dem Matrizenverfahren von Banachiewicz. Z. angew. Math. Mech. 29 (1949) 76–84. Z. Angew. Math. Mech. **30**, 130–132 (1950).

Comments by Bodewig on the paper by Zurmühl given in the title [see also these Rev. **10**, 743] and followed by further comments by Zurmühl.

Black, A. N. Further notes on the solution of algebraic linear simultaneous equations. Quart. J. Mech. Appl. Math. **2**, 321–324 (1949).

The author describes the method of Doolittle, which he calls the best method for linear equations. It has escaped his attention that this method is identical with Banachiewicz's method which again is identical to the abridged Gaussian method. [See Zurmühl, Z. Angew. Math. Mech. **29**, 76–84 (1949); these Rev. **10**, 743; also see the preceding review].

E. Bodewig (The Hague).

Quenouille, M. H. A further note on discriminatory analysis. Ann. Eugenics **15**, 11–14 (1949).

An iterative method is presented for finding the largest latent root and the associated latent vector $\mathbf{x}$ of the matrix $(\mathbf{A}-\lambda\mathbf{B})$, where $\mathbf{A}$ and $\mathbf{B}$ are positive definite. The method utilizes the fact that this vector maximizes $\lambda(\mathbf{x})=\mathbf{x}'\mathbf{A}\mathbf{x}/\mathbf{x}'\mathbf{B}\mathbf{x}$. A first approximation $\mathbf{x}_0$ is taken, then $\lambda(\mathbf{x}_0)$ is calculated.

For the second approximation, $\delta\mathbf{x}_0$ is chosen as

$$\delta\mathbf{x}_0 = a[\partial\lambda/\partial x_i]_{\mathbf{x}=\mathbf{x}_0} = a[2(\mathbf{A}-\lambda\mathbf{B})\mathbf{x}_0/\mathbf{x}_0'\mathbf{B}\mathbf{x}_0],$$

$a>0$. The second trial vector $\mathbf{x}_1 = \mathbf{x}_0 + \delta\mathbf{x}_0$ should give a higher value of $\lambda(\mathbf{x})$, to a first-term Taylor approximation, since

$$\lambda(\mathbf{x}_1) = \lambda(\mathbf{x}_0) + \delta\mathbf{x}_0'[\partial\lambda/\partial x_i]_{\mathbf{x}=\mathbf{x}_0}$$
$$= \lambda(\mathbf{x}_0) + a\sum_{i=1}^{m}[\partial\lambda/\partial x_i]^2_{\mathbf{x}=\mathbf{x}_0}.$$

Advice is given about the optimum choice of the factor a.

W. G. Cochran (Raleigh, N. C.).

Borkmann, Karl. Zu G. Opitz: Praktische Verfahren zur Lösung von Gleichungen vierten Grades. Z. angew. Math. Mech. 25/27 (1947), S. 171/173 und zu H. Blenk: Nomogramme für die Gleichung 4. Grades mit reellen oder komplexen Wurzeln. Z. angew. Math. Mech. 29 (1949), S. 58/61. Z. Angew. Math. Mech. 30, 132 (1950).

Comments on the two papers of the title [these Rev. 9, 405; 10, 486].

Jovanović, Milan K., and Ilić, Branislav. Graphical solution of some technologically important transcendental equations. Srpska Akad. Nauka. Zbornik Radova, Knj. I. Mašinskii Inst., Knj. 1, 43–53 (1949). (Serbian)

Varoli, Giuseppe. Sopra un metodo di iterazione per la risoluzione approssimata delle equazioni. Period. Mat. (4) 28, 44–51 (1950).

This is an expository paper on the solution of the equation $x=\varphi(x)$ by iteration. The author discusses the convergence of the process and gives an upper bound for the error committed by using an iterated value as an approximate solution.

E. Lukacs (Washington, D. C.).

Meriam, J. L. Procedure for the machine or numerical solution of ordinary linear differential equations for two-point linear boundary values. Math. Tables and Other Aids to Computation 3, 532–539 (1949).

The author gives an exposition of the general linear boundary value problem for the nth order linear equation

$$(1)\qquad f_0(x)y^{(n)}+f_1(x)y^{(n-1)}+\cdots+f_n(x)y=f(x)$$

on the interval $a\leqq x\leqq b$. The boundary conditions consist of n linear independent equations, each condition involving the values of y to $y^{(n-1)}$ at a particular point $x=x_k$; conditions involving the derivatives at the end points $x=a$ and $x=b$ only are considered in particular. If k conditions are prescribed at $x=a$ and $n-k$ at $x=b$, an $n-k$ parametric family (2) $\sum_{i=1}^{n-k}c_iy_i(x)$ of solutions is first constructed satisfying the k conditions at $x=a$ only. Here the $y_i(x)$ must be computed numerically as the solutions of $n-k$ independent initial value problems satisfying the k conditions at $x=a$. When (2) is substituted in the $n-k$ boundary conditions at $x=b$ the unknown c_i are finally determined from the resulting $n-k$ linear equations. This latter process is illustrated graphically in special cases and a suitable selection procedure for the initial conditions of the y_i discussed briefly.

H. O. Hartley (London).

★Ritchie, C. C. Forward integration of differential equations. Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 4, pp. 325–330.

The author gives a brief summary of a numerical method for solving boundary value problems for ordinary and partial differential equations, the procedure being explained for the equation $\Delta w=f(x,y)$, the region of integration $0\leqq x\leqq na$, $-\infty<y<\infty$, and the boundary conditions $w(0)=w(na)=0$. The method is a finite difference procedure using a grid of mesh-width a and the familiar approximation

$$(1)\qquad \Delta w\sim a^{-2}\{4w_0-w_1-w_2-w_3-w_4\},$$

where w_0 is the value of w at a grid point and the other w_i's are the four adjoining values. Starting at $x=0$, arbitrary values are taken on the line $x=a$ and further values are then recurrently computed from (1) for the lines $x=2a,\cdots,x=na$. The effect on the values obtained on $x=na$ of altering one of the arbitrary values on the line $x=a$ is computed in form of a matrix whose inversion permits the determination of the alterations on $x=a$ required to satisfy the prescribed boundary conditions on $x=na$. Generalisations to other boundary conditions are sketched. No attempt is made to account for the error in the approximation (1).

H. O. Hartley (London).

Fox, L. The numerical solution of elliptic differential equations when the boundary conditions involve a derivative. Philos. Trans. Roy. Soc. London. Ser. A. 242, 345–378 (1950).

The author describes numerical methods of solving boundary value problems for elliptic differential equations. The equations considered are for a two variable function $w(x,y)$ and are mainly of the 2d and 4th order, the regions of integration are usually simple, the boundaries either linear or curved, and the boundary conditions are of the forms $w=k_1$, $\partial w/\partial\nu=k_2$, $w+k_3\partial w/\partial\nu=k_4$, where the k_i are constants and $\partial/\partial\nu$ denotes differentiation normal to the boundary. The main method considered is iterative and consists of the following steps. (i) Cover the region of integration by a rectangular grid of suitable mesh. (ii) Replace all differentials in equation and boundary conditions by suitable finite difference approximations, resulting in a system of linear equations for the values of w at the grid points which is solved by relaxation methods. (iii) Based on the approximate solution w_0 obtained in (ii), compute the error terms for the finite difference approximations employed and add these to the "right hand sides" of equations and conditions and solve again until no further corrections are needed to the required numerical accuracy. The asset of the method is that its accuracy can be controlled. Numerical examples illustrate the practice of choosing the grid and appropriate finite difference approximations. The particular difficulties arising with normal derivatives on curved boundaries are solved by splitting into component derivatives in grid directions and setting up the boundary conditions at the intersections of the grid with the boundary, interpolation formulae being required for this purpose. In a subsidiary method the speed of the iteration convergence is forced by using Richardson's deferred approach to the limit.

H. O. Hartley (London).

Jung, F. Zur graphischen Behandlung des Tensors. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 157, 97–100 (1 plate) (1949).

The author considers the problem of graphically representing tensors. In particular, he is concerned with the stress and strain tensors of elasticity theory. For these tensors, the Culmann and Mohr representations are discussed. The work is illustrated by a graph.

N. Coburn (Ann Arbor, Mich.).

***Nörlund, N. E. Anwendung einer Funktionalgleichung in der Ausgleichungsrechnung zur Bestimmung der Gewichte der Unbekannten.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 325–337, Rome, 1943.

Mittmann, Otfrid M. J. Ausgleichsrechnung mit einem Operator. Math. Nachr. 3, 102–106 (1949).

In connection with the adjustment of observations by least squares the author defines a linear operator, develops its rules of calculation, applies it to the problem of representing empirical values by a constant, or by a function linear in certain parameters, and also applies it to finding the mean error of the adjusted results. *W. E. Milne.*

Ridenour, Louis N. High speed digital computers. An elementary survey of present developments and future trends. J. Appl. Phys. 21, 263–270 (1950).

Marshall, Byron O., Jr. The electronic isograph for roots of polynomials. J. Appl. Phys. 21, 307–312 (1950).

Grivet, P., et Rocard, Yves. La réaction dans les chaînes et réseaux d'analogie électrique. Revue Sci. 87, 85 (1949).

An electrical circuit is given which solves the differential equation

$$\frac{d^2V}{dx^2}+a(x)\frac{d}{dx}\frac{1}{a(x)}\frac{dV}{dx}+b(x)V+c(x)=0.$$

G. Kron (Schenectady, N. Y.).

Wilson, L. H., and Miles, A. J. Application of the membrane analogy to the solution of heat-conduction problems. J. Appl. Phys. 21, 532–535 (1950).

ASTRONOMY

Zagrebin, D. V. Concerning the accuracy of the Stokes formula. Akad. Nauk SSSR. Bull. Inst. Teoret. Astr. 4, no. 3(56), 134–141 (1949). (Russian)

The formula of Stokes has been used for nearly 100 years in connection with the problem of deriving the departures of the geoid from a sphere. The author had derived a set of formulae in which the comparison is made against an ellipsoidal equipotential surface. His original expression gave an error which is comparable to that of the classical formula. He has now developed a new set of expressions, involving several functions which he has tabulated, which give the radii of curvature, M and N, of the meridian and the first vertical, retaining all terms of the order of i^2 where i is the second eccentricity of the ellipsoid. The new method is applied to the computation of departures of an idealized geoid from an ellipsoid of comparison, for both of which he gives the necessary data. A table lists the true values of N, those computed by the new formula (N_z) for different values of λ and φ, the differences $N-N_z$, where N_s is the value obtained from the classical Stokes formula, the differences $N-N_s$, and the quantity $\zeta\alpha$ which measures the precision of the Stokes formula. The author concludes that the precision obtained by the use of his new formulae is at least three times that of the old procedure.

O. Struve (Williams Bay, Wis.).

Prasad, Chandrika. On a general theorem on rotating masses by Jeans. Proc. Benares Math. Soc. (N.S.) 9, 33–35 (1947).

In considering a generalization of a theorem by Poincaré concerning the upper limit of the angular velocity of a rotating fluid mass, Jeans [Astronomy and Cosmogony, Cambridge University Press, 1929, p. 264] has omitted terms due to Coriolis forces. These are proportional to velocities which are assumed to be small and superposed on the uniform rotation. Starting with the correct form of the equations of motion the author concludes that "the above investigation shows that Poincaré's theorem cannot be generalised for fluids rotating approximately as rigid bodies." Nevertheless, it seems to the reviewer that the above generalisation is possible, but must be given in a more precise form.

W. S. Jardetzky (New York, N. Y.).

Jeffreys, Harold. Dynamic effects of a liquid core. Monthly Not. Roy. Astr. Soc. 109, 670–687 (1949).

Le calcul des marées terrestres fait par Herglotz pour une coque et un noyau homogènes élastiques a été appliqué par Jeffreys en 1926 et Rosenhead en 1929 au cas d'un noyau liquide, en annulant simplement la rigidité. En réalité les effets d'inertie ne sont pas négligeables pour les marées qui déplacent l'axe principal d'inertie. Comme l'a remarqué Poincaré, les vitesses sont alors des fonctions linéaires des coordonnées. Jeffreys résout le problème en employant les coefficients de ces fonctions comme coordonnées de Lagrange. Numériquement, la période d'Euler pour une coque indéformable et un noyau liquide étant plus faible que pour une terre homogène, on pouvait craindre de voir disparaître l'accord avec l'observation constaté anterieurement. Mais l'élasticité de la coque diminue les mouvements dans le noyau liquide, et il suffit de réduire de 5% la rigidité supposée de la coque. Par contre le désaccord entre la nutation lunaire observée et celle que l'on calcule à partir de l'inégalité mensuelle n'est pas réduit sensiblement par la nouvelle théorie, même avec une correction tenant compte de l'obliquité de l'écliptique. *J. Coulomb* (Paris).

Kühn, W. Über den inneren Aufbau eines mit konstanter Winkelgeschwindigkeit rotierenden polytropen Sterns. Astr. Nachr. 277, 97–111 (1949).

The paper develops a method of approximation which the author suggests for use in studying the internal constitution of a polytropic star rotating with constant angular velocity. The method consists in replacing surfaces of equal pressure and density by rotationally symmetrical ellipsoids with varying eccentricity throughout the star. It is not possible to make ellipsoids satisfy the equations exactly. They are therefore made to satisfy the equations only approximately. The method appears troublesome, and the author does not go through with the calculations, but gives numerical examples only of models with constant ellipticity through the star. *G. Randers* (Oslo).

Kurth, Rudolf. Über Sternsysteme zeitlich oder räumlich veränderlicher Dichte. Z. Astrophys. 26, 100–136 (1949).

In this paper the solutions of Liouville's equation given by Chandrasekhar and Schürer [cf. Schürer, Astr. Nachr.

273, 230–242 (1943); these Rev. 6, 244] for nonstationary stellar systems are considered in conjunction with Poisson's equation. It is shown that in general Poisson's equation cannot be satisfied unless the system is either stationary or homogeneous. In the stationary case the author proposes a generalization of Schwarzschild's ellipsoidal distribution and indicates how with this generalization Poisson's equation can be solved by expanding the various quantities in terms of a suitable parameter; the convergence of these series is also examined. *S. Chandrasekhar* (Williams Bay, Wis.).

Kurth, Rudolf. Zur Dynamik instationärer Sternsysteme. Z. Astrophys. 26, 168–175 (1949).

In this paper the author gives another proof of the theorem stated in the paper reviewed above. Considering the cases of spherical and axial symmetries the author further suggests that solutions for nonstationary systems derived from Liouville's equation necessarily have zero density if the systems are assumed to be confined to a finite volume of space and satisfy certain other restrictions of differentiability and continuity. *S. Chandrasekhar.*

RELATIVITY

Soudan, R. La théorie de la relativité et l'électromagnétisme. Arch. Sci. Soc. Phys. Hist. Nat. Genève 3, 5–16 (1950).

The author shows that if the metric tensor is assumed to depend on e/m, then, if certain approximations are made, the equations of the geodesics in space-time reduce to those describing the motion of a charged particle in an electromagnetic field without gravitation being present. The vector potential of this field is given in terms of the gravitational tensor by the equations $2(e/m)\varphi_4=g_{44}-1$, $(e/m)\varphi_i=g_{4i}+1$. It is verified that under certain assumptions on the form of the stress energy tensor, the Einstein field equations become those for the Maxwell field. However, the author does not discuss the difference in transformation properties of the four-vector potential and the gravitational potentials which are equated in a particular class of coordinate systems.
A. H. Taub (Urbana, Ill.).

Imaeda, K. Linearization of Minkowski space and five-dimensional space. Progress Theoret. Physics 5, 133–134 (1950).

The author uses the components of the electromagnetic field to define a biquaternion and writes Maxwell's equations in terms of such quantities. However, the transformation properties of these equations are not discussed. A hypercomplex number system with five basic elements is used to deal with the Proca equations. *A. H. Taub.*

Novobátzky, K. F. Einheitliche Feldtheorie in vier Dimensionen. Hungarica Acta Physica 1, no. 5, 1–6 (1949).

The author defines a parallel displacement in a Riemannian 4-space by the relation

$$dv^i=-\overset{0}{\Gamma}{}^i_{jk}v^jdx^k-F^i{}_jv^jds$$

where $\overset{0}{\Gamma}{}^i_{jk}$ is the Christoffel symbol and F_{ij} is an antisymmetric tensor and ds is the distance from A $(=x^i)$ to B $(=x^i+dx^i)$. He then shows that the difference of two parallel vector fields will be independent of the point at which the comparison is made if $ds_{AB}+ds_{BA}=0$. This requires an ordering of points in space-time which he proposes to set up in terms of the values of four algebraic invariants built up from F_{ij}, g_{ij} and the "vector potential" ϕ_i. It is pointed out that the Maxwell-Lorentz pondermotive force equations may be expressed in terms of this parallelism. However, the author does not interpret the constant e/m occurring in that equation; nor does he discuss the field equations. *A. H. Taub* (Urbana, Ill.).

Podolanski, J. Unified field theory in six dimensions. Proc. Roy. Soc. London. Ser. A. 201, 234–260 (1950).

The interpretation of the fifteen independent Dirac matrices as rotation operators suggests this attempt to formulate a unified field theory using a six-dimensional space. In contrast to the Kaluza-Klein five-dimensional theory [T. Kaluza, S.-B. Preuss. Akad. Wiss. 1921, 966–972; O. Klein, Z. Physik 37, 895–906 (1926)] a projective interpretation of the coordinates is not attempted. Instead a geometric condition is postulated so that physical phenomena are independent of the two extra coordinates, i.e., each world point corresponds to a two-dimensional sheet. Requiring the path of a material particle to be a six-dimensional geodesic gives, when projected into the four real dimensions, equations of motion of the usual form for a particle subject to the Lorentz forces of two Maxwell type fields. The field quantities are found to obey equations which are the customary gravitational equations plus Maxwell equations for fields of positive and negative definite energies. Requiring the material particle to follow a null geodesic makes the theory conformally invariant and yields a relation between the mass and the charges with respect to the two Maxwell fields. It is stated, without proof, that by relaxing the geometric axiom and considering quantum theory the second Maxwell field may be replaced by a finite range Proca field capable of bearing charge. This would result in an automatic elimination of classical self-energy difficulties by means of compensation. *K. M. Case* (Princeton, N. J.).

Fihtengol'c, I. G. The Lagrangian form of the equations of motion of Einstein's theory of gravitation in second approximation. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 20, 233–242 (1950). (Russian)

Using methods due to Fock, Petrova [Zhurnal Eksper. Teoret. Fiz. 19, 989–999 (1949); these Rev. 11, 467] found equations in harmonic coordinates which she interpreted as the next approximation after Newton's to the equations of motion, in Einstein's theory, of a system of n bodies with finite mass. The author shows that Petrova's equations may be written in Lagrangian and Hamiltonian form and explicitly displays L and H. If in the expression for the latter, the gravitational constant γ is set equal to zero, H reduces to $\frac{1}{2}mv^2+\frac{3}{8}mv^4/c^2$, thus agreeing with the expression for kinetic energy in the special theory of relativity. Equations which are strict analogues of the classical laws of linear and angular momentum follow almost immediately from the Lagrangian form of Petrova's equations. By defining the mass of the ith particle as

$$M_{(i)}=m_{(i)}[1+\tfrac{1}{2}v^2{}_{(i)}/c^2-\tfrac{1}{2}c^{-2}\sum_{k\neq i}\gamma m_{(k)}/r_{(ik)}],$$

where $m_{(k)}$ is the coefficient of $\ddot{x}^j{}_{(k)}$ in Petrova's equations, and setting $K^j=\sum_k M_{(k)}x^j{}_{(k)}$, the author finds that $\dot{K}^j$ are constants. In other words, with respect to Fock's harmonic coordinates, the point (K^j) moves like the classical center of inertia. *A. J. Coleman* (Toronto, Ont.).

Robertson, H. P. The geometries of the thermal and gravitational fields. Amer. Math. Monthly 57, 232–245 (1950).

The author's purpose is to give "a study of the physical geometry of the gravitational and of the thermal fields, and of the reasons for the success of the former as opposed to the latter, as a basis for a tenable physical theory." The discussion of gravitation is given in terms of a four-dimensional scalar theory but the main ideas of Einstein's general theory are stressed. The results of this theory regarding the motion of test particles and the behavior of light rays are stated and compared to that of Einstein's theory and differ from that theory. However the author's purpose is not to give a physical theory of gravitation, but to discuss the geometry of a theory of the gravitational field. He also discusses the geometry of the thermal field in terms of measurements made by rods after they have been allowed to come into thermal equilibrium with a heated medium. It is shown that in the latter case the geometry depends on the material composing the measuring rods whereas in the gravitational case the geometry is universal because of the equivalence between inertial and gravitational mass.

A. H. Taub (Urbana, Ill.).

MECHANICS

Federhofer, K. Zur graphischen Kinetostatik ebener Getriebe. Österreich. Ing.-Arch. 4, 130–135 (1950).

This is an extension of the work of the author and others on the graphical solution of plane kineto-static problems. The displacement diagram is used to determine forces by a balance of the moments on the Joukowski lever [see S. Timoshenko and D. H. Young, Theory of Structures, McGraw-Hill, London-New York, 1945, p. 42] as well as the velocity and acceleration diagrams. As an example of the method, it is applied to the determination of the forces on the camshaft of an internal combustion engine.

M. Goldberg (Washington, D. C.).

Bottema, O. On the kinematic representation of Beth and its application to the central connecting-rod motion. Euclides, Groningen 25, 253–256 (1950). (Dutch)

Carter, B. C. Analytical treatment of linked levers and allied mechanisms. J. Roy. Aeronaut. Soc. 54, 247–252 (1950).

Two cranks, which rotate about nonintersecting axes, are joined by a ball-ended connecting-rod. The relative rotations and velocities of the cranks are derived. A similar derivation is obtained for shafts joined by Hooke's universal joints.

M. Goldberg (Washington, D. C.).

Tolotti, Carlo. Sul moto impulsivo di un sistema olonomo soggetto simultaneamente a più vincoli unilaterali. Ann. Mat. Pura Appl. (4) 29, 251–257 (1949).

In elementary mechanics, motions involving two colliding bodies are studied with the help of a hypothesis due to Newton to the effect that the ratio of the relative velocity just after collision to the relative velocity just before collision is a constant (coefficient of restitution). The author generalizes the resulting equations so as to apply to general holonomic dynamical systems subjected to several simultaneous one-sided constraints. The proof of the final equations is based on the Newtonian hypothesis as well as on assumptions about the types of admissible one-sided constraints.

D. C. Lewis (Baltimore, Md.).

Agostinelli, Cataldo. Sulla stabilità di un particolare moto di precessione regolare di un solido pesante asimmetrico. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 107, 193–203 (1949).

The author studies the characteristic exponents of a particular periodic solution of the equations of motion for a heavy rigid body about a fixed point in case the center of gravity is on the axis of a circular section of the ellipsoid of inertia. It turns out that the motion is generally unstable except when the moments of inertia satisfy certain relations.

D. C. Lewis (Baltimore, Md.).

Agostinelli, Cataldo. Sul moto intorno a un punto fisso di un corpo rigido pesante il cui baricentro appartiene all'asse di uno dei piani ciclici dell'ellissoide d'inerzia. Ann. Mat. Pura Appl. (4) 30, 211–224 (1949).

The author considers the problem of the motion of a heavy rigid body about a fixed point in the case where the center of gravity lies on the axis of one of the circular sections of the ellipsoid of inertia. He proves the existence of certain motions, depending on five constants of integration, which are expressed parametrically by means of certain power series. The recursion formulas for the coefficients of these series are explicitly given. When the moments of inertia satisfy a certain relationship, these series reduce to polynomials, at least for certain values of the integration constants.

D. C. Lewis (Baltimore, Md.).

Myasnikov, P. V. On the representation of the motion of a rigid body about a fixed point by means of governing surfaces. Vestnik Moskov. Univ. 4, no. 10, 19–27 (1949). (Russian)

This carelessly edited paper deals with gyrations (M) for which the angular-momentum vector K (hence also the resultant torque) has a fixed direction. The simple basic fact is that for motions (M) the ratio T/K^2 is constant (T is kinetic energy). If Ox, Oy, Oz are the inertial axes for the fixed point O, the "governing" (entrained) ellipsoid $x^2/A+y^2/B+z^2/C=2T/K^2$ (A, B, C are the principal moments of inertia at O) contains the tip of the unit vector coaxial with K. Further, the distance from O of the tangent plane at the ellipsoid's intersection with the angular-velocity vector ω is $\cos(\omega, K)$. The author seems to want next to show that every gyration can be significantly associated with a motion of type (M). This the reviewer failed to grasp. The association is illustrated by means of a highly invented example.

A. W. Wundheiler (Chicago, Ill.).

Grammel, R. Zur Berechnung der Poinsotbewegung. Ing.-Arch. 18, 53–59 (1950).

The differential equations for the motion of a rigid body about a fixed point under the influence of no external forces can be integrated by quadratures and also admit the geometric interpretation of Poinsot. The author modifies the well-known solution in terms of elliptic functions so as to obtain approximate formulas more suitable for computational purposes. This is done by discarding all but the first terms in the Fourier series of certain theta functions.

D. C. Lewis (Baltimore, Md.).

Manarini, Mario. Sulla stabilizzazione di uno stato di equilibrio mediante azioni girostatiche. Boll. Un. Mat. Ital. (3) 5, 56–63 (1950).

Consider a system of differential equations of the form $\ddot{M}+G\dot{M}=\text{grad } U(M)$, where M is a vector in n-dimensional space, G is a skew-symmetric matrix, U is a scalar function of M, and the dots represent differentiation with respect to t. By introducing suitable rotating coordinates, the author eliminates the term in $G\dot{M}$ and thus reduces the study of stability of equilibrium to the known Lagrange-Dirichlet theory. *D. C. Lewis* (Baltimore, Md.).

Negri, Domenico. Su un notevole integrale primo che si incontra in cosmogonia. Atti Sem. Mat. Fis. Univ. Modena 3, 223–226 (1949).

Dans cette note on part d'une loi élémentaire de force dépendant d'une manière quelconque de la distance de deux points matériels qui s'attirent et de la dérivée temporelle de la distance même et on envisage les équations différentielles du mouvement d'un point matériel qui est attiré par un corps en rotation. Alors les équations admettent une intégrale première où interviennent le moment cinétique du corps par rapport à son centre de masse et la vitesse aréale du point. Cette intégrale contient en particulier celle que Agostinelli a établie [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 166–186 (1940); ces Rev. 8, 410] à propos d'une correction à la loi de Newton proposée par Armellini. *G. Lampariello* (Messine).

Egerváry, E. Sur une nouvelle solution particulière du problème des trois corps. Comment. Math. Helv. 24, 1–3 (1950).

L'auteur envisage le problème du mouvement de trois corps qui s'attirent en raison directe du cube des distances mutuelles. Lorsque les masses des corps sont égales, il trouve une intégrale première, indépendante des intégrales classiques. Cette intégrale nouvelle permet de déterminer une famille de solutions particulières dans le cas du mouvement plan. *G. Lampariello* (Messine).

Nardini, Renato. Su un sistema dissipativo ad n gradi di libertà. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 7 (1949), 224–227 (1950).

The purpose of this paper is to find conditions sufficient for the stability of linear dissipative system in which the potential energy depends explicitly on the time. The method involves setting up and differentiating a certain positive definite quadratic form in the coordinates and the velocities. *D. C. Lewis* (Baltimore, Md.).

Gallissot, François. Sur une forme des équations du mouvement d'un système matériel à liaisons holonomes ou non avec ou sans frottement. C. R. Acad. Sci. Paris 230, 511–512 (1950).

Using the general dynamical equations of Appell, a theorem is obtained relating to one-sided constraints. These occur when two solids of the system come into contact. For two such solids, a "velocity of contingence" is defined of one solid relative to the other. It is found that the time derivatives of the components of this velocity of contingence break up into two terms in such a way that it is possible to analyze the type of contact, e.g., contacts which persist, contacts which cease, shocks, rolling motions, and sliding motions. *D. C. Lewis* (Baltimore, Md.).

Gallissot, François. Sur la discussion des éventualités dans un système à k contacts avec ou sans frottement. C. R. Acad. Sci. Paris 230, 611–612 (1950).

Consequences of the general theorem of the paper reviewed above. The possible motions subsequent to a given contact are analyzed by means of the signature of a certain quadratic form. *D. C. Lewis* (Baltimore, Md.).

Pailloux, Henri. Sur certains systèmes non holonomes. C. R. Acad. Sci. Paris 230, 1501–1504 (1950).

The systems referred to are those whose constraint equations are nonlinear in the velocities and have the further property of being "perfect," i.e., the forces of constraint do no work in any virtual displacement compatible with the restraints. In such cases the equations of motion are much simpler than in the general case. The linear case is excluded apparently because a linear perfect system is essentially holonomic. *D. C. Lewis* (Baltimore, Md.).

Pailloux, Henri. Extension de la notion de paramètre de Lagrange. C. R. Acad. Sci. Paris 230, 1136–1138 (1950).

L'auteur propose ici l'emploi de fonctions-paramètres pour l'étude des mouvements d'un système dynamique dont les configurations dépendent de fonctions arbitraires. Dans cette note il envisage, pour fixer les idées, le cas d'un système matériel qui dépend d'un seul fonction arbitraire φ d'un point μ variable dans un domaine Δ fixe de l'espace. Soit M un point quelconque du système, sa variation $\delta\overrightarrow{M}$ peut être représentée par une fonctionnelle

$$\delta\overrightarrow{M}=\int_{\Delta_1}(\partial\overrightarrow{M}/\partial\varphi)\delta\varphi(\mu_1)d\tau_1=\int_{\Delta_1}\mathfrak{M}[\varphi(\mu),\mu_1]\delta\varphi(\mu_1)d\tau_1$$

en désignant par Δ_1 le même domaine Δ décrit par le point μ_1 et par $d\tau_1$ l'élément de volume au point μ_1. Si F désigne la résultante des forces agissant sur M, le travail virtuel est donné par la fonctionnelle

$$\delta\mathcal{T}=\int_{\Delta_1}\phi[\varphi(\mu),\mu_1]\delta\varphi(\mu_1)d\tau_1$$

avec $\phi=\sum F\cdot\int\mathfrak{M}[\varphi(\mu),\mu_1]d\tau$, tandis que la variation de la demiforce vive T est représentée par

$$\delta T=\int_{\Delta_1}(\partial T/\partial\varphi)\delta\varphi(\mu_1)d\tau_1+\int_{\Delta_1}(\partial T/\partial\varphi')\delta\varphi'(\mu_1)d\tau_1,$$

$'=\partial/\partial t$. Cela posé, si l'on part du principe d'Hamilton, on trouve d'après le procédé classique du calcul des variations, les équations du mouvement du système sous la forme

$$\frac{\partial T}{\partial\varphi}-\frac{d}{dt}\frac{\partial T}{\partial\varphi'}+\phi=0,$$

où figurent des dérivées fonctionnelles à la place des dérivées partielles ordinaires. *G. Lampariello* (Messine).

Sonntag, G. Berechnung des Spannungszustandes und Schlupfes beim Rollen deformierbarer Kugeln. Z. Angew. Math. Mech. 30, 73–83 (1950). (German. English, French, and Russian summaries)

Calculations of stresses, slipping, and friction for rolling spheres are made. They neglect change of shape of cross-section, but are shown to be good approximations by use of rigorously obtained upper and lower bounds. *P. Franklin* (Cambridge, Mass.).

Gantmacher, F. R., and Levin, L. M. Equations of motion of a rocket. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1255**, 21 pp. (1950).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 301–312 (1947); these Rev. **9**, 162.

McShane, E. J. The differentials of certain functionals in exterior ballistics. Duke Math. J. **17**, 115–134 (1950).

If a projectile is launched with given initial conditions, its range, time of flight, etc., are functionals of the range-wind, cross-wind, density, temperature, and of other functions of position and velocity. In exterior ballistics it is customary first of all to compute the trajectories under certain relatively simple "normal" conditions regarding wind (identically zero), densities of the air at different altitudes, etc., and then to consider corrections which must be made to such trajectories to account for abnormal disturbances which are relatively small. Effects of departures from "normal" conditions are approximated by the differential of the functional.

The range is the quantity of greatest interest to the ballistician in most problems of artillery fire. If one restricts attention to the effect of wind on range, the range is a functional of the wind. G. A. Bliss [Trans. Amer. Math. Soc. **21**, 79–92, 93–106 (1920)] proved that the differential of this functional exists if the space of wind functions is normed by $\|w\| = \max [\sup |w(y)|, |dw(y)/dy|]$. In the paper under review the author points out that with certain disturbances (for example the wind, temperature, density) the inclusion, in the definition of the norm, of the derivatives with respect to the altitude y is undesirable for physical reasons. In order to avoid the consideration of two separate classes of disturbances, the author defines the norm of a disturbing function to be the greatest of the maxima of the absolute values of the function itself and of its partial derivatives with respect to all variables except the altitude y. The norm being defined, it is shown (1) that with given "normal" conditions and given disturbances the equations of motion have solutions uniquely determined by the initial conditions and that the solutions are continuous functions of the disturbances, provided that certain conditions are satisfied; (2) given an original trajectory, disturbances from the conditions on this trajectory do produce differentiable effects on range, etc., provided that the functions determining the original trajectory are continuously differentiable with respect to all variables, including y, near the summit (in practice this assumption is harmless, for stronger continuity properties hold for the "normal" trajectories of ballistics); (3) the difference between differential effect and actual change is an infinitesimal of second order in the norm if the trajectory is everywhere ascending or everywhere descending. *E. Leimanis* (Vancouver, B. C.).

Hydrodynamics, Aerodynamics, Acoustics

Hölder, Ernst. Über die Variationsprinzipe der Mechanik der Kontinua. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **97**, no. 2, 13 pp. (1950).

In this paper the motion of a continuous medium is described by the three functions $b^i = b^i(x^\beta, t)$, where x^β are Eulerian coordinates referred to axes rotating with angular velocity $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$, t the time, and b^i Lagrangian coordinates. In terms of these functions, the density is $\rho = \rho_0 \det(p_\beta{}^i)$, with $\rho_0 = \rho_0(b^i)$ and $p_\beta{}^i = \partial b^i/\partial x^\beta$, and the velocity is $u^\gamma = -a_i{}^\gamma p_0{}^i$ where $a_i{}^\alpha p_\beta{}^i = \delta_\beta{}^\alpha$ and $p_0{}^i = \partial b^i/\partial t$ $(x^0 = t)$. The medium is assumed to possess internal energy per unit volume $\Omega(p_\beta{}^i, b^i, s)$ where s is entropy-density, a function of b^i (adiabatic hypothesis). The central hypothesis is a Hamiltonian principle (a generalisation of one due to Clebsch) which reads $\delta \int F(t, x^\beta; b^i, p_\beta{}^i)dxdt = 0$, where $F = \frac{1}{2}\rho u_\gamma u^\gamma + \rho\omega_{\delta\gamma} x^\delta u^\gamma - \Omega - \rho U$, U being potential energy of body forces per unit mass. With $\mu, \nu = 0, 1, 2, 3$, and other suffixes 1, 2, 3, the author defines an energy-momentum tensor $U_\mu{}^\nu = F\delta_\mu{}^\nu - p_\mu{}^j \pi_j{}^\nu$, where $\pi_j{}^\nu = \partial F/\partial p_\nu{}^j$, and establishes the identity $dU_\mu{}^\nu/dx^\nu - F_{x^\mu} = p_\mu{}^j(F_{b^j} - d\pi_j{}^\nu/dx^\nu)$. Then the Euler-Lagrange equations of the variational principle imply the four conservation laws $dU_\mu{}^\nu/dx^\nu - F_{x^\mu} = 0$. The equation of continuity is satisfied and the conservation laws for $\mu = 1, 2, 3$ give the equations of motion

$$\rho(du_\alpha/dt + 2\omega_{\beta\alpha}u^\beta) = (d/dx^\beta)(\Omega\delta_\alpha{}^\beta - p_\alpha{}^i \partial\Omega/\partial p_\beta{}^i) - \rho\partial U/\partial x^\alpha.$$

Specialisations are made to a gas, with $\Omega = f(\rho, s, b^i)$, and in particular to its steady streaming. *J. L. Synge.*

***Anđelić, Tatomir P. Osnovi mehanike neprekidnih sredina. [Foundations of the Mechanics of Continuous Media].** Naučna Knjiga, Belgrade, 1950. vii+231 pp.

This book is primarily an exposition of the mathematical foundations of the theory of elasticity and fluid mechanics. Emphasis is on the derivation of the fundamental equations and general theorems, almost no consideration being given to problems associated with special geometrical configurations. The book is divided as follows: Introductory chapter on dyadics, affinors, and tensors; chapter I. Strain and stress in continuous media; chapter II. Theory of elasticity; chapter III. Hydromechanics. *J. V. Wehausen.*

Kasterin, N. P. Resolution of Felix Klein's aerodynamic paradox. Vestnik Moskov. Univ. **4**, no. 10, 45–51 (1949). (Russian)

This paradox predicts an infinite kinetic energy, in cylindrical, irrotational, adiabatic and steady flow around a vortex column, for every fluid zone between two transverse planes. The derivation is based on the radial momentum equation $\rho V^2/r = dp/d\rho$ (persistently and confusingly misprinted with a minus sign) and the zero-curl condition Vr = constant. The paper's resolution is based on the repudiation of the usual momentum equation inside of sufficiently small fluid elements which the author assumes to be engaged in rigid rotation. He concludes that the centripetal acceleration points away from the vortex column, and that the Euler equations of motion should be changed to read $u_t - u\dot{u} - v\dot{v} = -p_x/\rho$, etc. Both in this argument and in an experimental study (said to confirm the author's theory) there is an obvious confusion of the angular velocity of gyration and the azimuthal velocity V/r. No refutation of other imaginable resolutions is presented. The paper has been posthumously reconstructed by the author's associates [Timiryazev et al.]. *A. W. Wundheiler* (Chicago, Ill.).

de Kármán, Théodore. Accelerated flow of an incompressible fluid with wake formation. Ann. Mat. Pura Appl. (4) **29**, 247–249 (1949).

The author determines an example of a symmetrical unsteady flow with finite closed wake (of constant pressure) behind a flat plate. The flow problem involved differs from the well-known steady free boundary problem in that the nonstationary form of Bernoulli's equation replaces the condition of constant flow speed as boundary condition on

the free streamlines. In the author's example the complex velocity potential $F(z)$ is of the form, $F(z)=U(t)f(z)$, ($U(t)$ = velocity at ∞); the flow therefore has the property that the wake is of constant shape. The flow is given in parametric form by

$$\left.\begin{aligned} F&=UAh[1+\tfrac{1}{4}(\zeta+(1/\zeta))],\\ \frac{dF}{dz}&=U(1+\zeta)[(1-\zeta/k)/(1-k\zeta)]^{\frac{1}{2}}, \end{aligned}\right\} |\zeta|\leqq 1,\ \Im\zeta\geqq 0,$$

where A and k are known numerical constants, and h is the half width of the plate. In order to satisfy the conditions on the flow problem $U(t)$ must be of the special form $U(t)=U(0)/(1-(U(0)t/hkA))$.

[Reviewer's note: It is possible to derive the entire set of flows with finite closed wake, and in the process to show that the above example is one of a class of flows having a wake of constant shape; this is contrary to the author's remark that his example is probably the only flow with this property.] *D. Gilbarg* (Bloomington, Ind.).

Samoĭlovič, G. S. Calculation of hydrodynamical lattices. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 121–138 (1950). (Russian)

The author gives a general method of calculating the flow around a lattice of profiles when the flow around one profile is known. Consider a region G exterior to the given lattice of unit circles with gap t in the z-plane and the region g exterior to the given lattice of profiles L with gap t_1 in the ζ-plane. The gap t_1 and the form of the profile L are assumed to be arbitrary. The method consists in constructing a function which maps conformally the region G on the region g, assuming that the function which maps the region exterior to the unit circle about the origin into the region exterior to the single profile L is known. Let the infinities in both planes correspond. Then the mapping function can be written in the form $\zeta=az+P(z)$, where $P(z)$ has the following properties: (i) it is a periodic function of z with period it; (ii) it remains bounded for $z\rightarrow\infty$; and (iii) in any neighborhood of the origin, and therefore, because of the periodicity of $P(z)$ in the neighborhood of any circle, the expansion $P(z)=\sum_{n=-\infty}^{\infty}a_nz^n$ is valid. The author obtains for ζ the expression

$$(1)\qquad \zeta=az+\frac{\pi}{t}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}a_{-(n+1)}\frac{d^n}{dz^n}\coth\frac{\pi z}{t},$$

where $a_{-n}=(2\pi i)^{-1}\int_c P(w)w^{n-1}dw$ and where the integration is performed along a unit circle about the origin. The series (1) plays for the calculation of hydrodynamical lattices the same role as the Laurent series in mapping the region exterior to the unit circle around the origin on the exterior of a single profile L.

For the complex potential $F(\zeta)$ of the lattice flow the expansion

$$(2)\quad F(\zeta)=\left(a\bar{W}-\frac{\lambda\Gamma}{2\pi i}\right)z(\zeta)+\frac{\Gamma}{2\pi i}\ln\frac{1}{\lambda}\sinh\lambda z(\zeta)$$
$$+\lambda\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}a_{-(n+1)}\frac{d^n}{dz^n}\coth\lambda z(\zeta)$$

is obtained, where Γ is the circulation around a single profile L. In the limiting case when $\lambda=\pi/t\rightarrow 0$, i.e., when $t\rightarrow\infty$, the mapping series (1) goes over into the corresponding Laurent series and (2) into the well-known expression for the complex potential of the flow past a single profile L. A method of calculating the coefficients a and a_{-n} of the series (1) for given λ in terms of the coefficients c_1 and c_{-n} of the corresponding Laurent series is then given. As examples the calculation of potential flow through a lattice of circles and through the blades of a steam turbine is given.

E. Leimanis (Vancouver, B. C.).

Berker, Ratip. Inégalité vérifiée par l'énergie cinétique d'un fluide visqueux incompressible occupant un domaine spatial borné. Bull. Tech. Univ. Istanbul 2, 41–50 (1949). (French. Turkish summary)

This is a detailed version of a paper which appeared previously [C. R. Acad. Sci. Paris 228, 1327–1329 (1949); these Rev. 10, 636]. *Y. H. Kuo* (Ithaca, N. Y.).

Illingworth, C. R. Some solutions of the equations of flow of a viscous compressible fluid. Proc. Cambridge Philos. Soc. 46, 469–478 (1950).

In searching for simple solutions of the Navier-Stokes equations for a viscous compressible fluid, the author considers the following problems: (1) simple shear motion, (2) flow past a porous flat plate, (3) simple shear motion between rotating cylinders, (4) circulatory flow round a circular cylinder with suction at the surface, and (5) the flow produced by a rotating disk. In the last problem the equations are shown to be reducible to ordinary differential equations for which no solutions are given.

Y. H. Kuo (Ithaca, N. Y.).

Davies, C. N. Viscous flow transverse to a circular cylinder. Proc. Phys. Soc. Sect. B. 63, 288–296 (1950).

This is a slight improvement of Lamb's well-known solution [Hydrodynamics, 6th ed., Cambridge University Press, 1932, pp. 614–616] of a viscous flow past a circular cylinder for small Reynolds number. *Y. H. Kuo* (Ithaca, N. Y.).

Preston, J. H. The steady circulatory flow about a circular cylinder with uniformly distributed suction at the surface. Aeronaut. Quart. 1, 319–338 (1950).

The general solution of the equations of steady two-dimensional motion of a viscous incompressible fluid, in which the velocity field depends only on an angular polar coordinate, is obtained, and used to determine the flow due to the rotation, in unlimited fluid, of a porous cylinder of radius a with a uniform suction velocity w_0 through the surface and an arbitrary circulation at infinity. This exists whenever the Reynolds number $R=w_0a/\nu$ exceeds 2. The vorticity varies as the inverse Rth power of the distance from the cylinder, and is therefore confined near the cylinder by the action of suction at a Reynolds number exceeding 2. (In the absence of suction the vorticity is known to diffuse indefinitely, and thus is uniform in the theoretical steady state.) Full details of the solutions (e.g., stress distributions and required suction power) are given, as well as an extended solution with an added stream parallel to the axis of the cylinder, and a discussion of the relation of the solution to practical problems of the use of rounded porous wings, around which the circulation is set up by vortices cast-off from a continuously adjustable, and withdrawable, rear flap, as suggested by B. Thwaites [J. Roy. Aeronaut. Soc. 52, 117–124 (1948)]. *M. J. Lighthill* (Manchester).

Thwaites, B. Note on the circulatory flow about a circular cylinder through which the normal velocity is large. Quart. J. Mech. Appl. Math. 3, 74–79 (1950).

This paper considers a symmetrical flow of viscous fluid about a fixed infinite circular cylinder with uniform suction.

Solutions in the form of asymptotic series in the Reynolds number are obtained. For both steady and unsteady flows, the circulation at infinity can remain constant if the Reynolds number or the suction velocity exceeds a minimum value. It is shown that the circulation increases rapidly with the radius r to the asymptotic value for large Reynolds number. A comparison with boundary-layer theory is therefore made. This investigation was suggested by recent speculation of the author [J. Roy. Aeronaut. Soc. **52**, 117–124 (1948)] regarding the production of lift independent of incidence. *Y. H. Kuo* (Ithaca, N. Y.).

Trilling, Leon. The incompressible boundary layer with pressure gradient and suction. J. Aeronaut. Sci. **17**, 335–342 (1950).

Consider a laminar boundary layer in plane flow; let $u(x, y)$, $v(x, y)$, $p(x)$ denote the usual velocity components and pressure, all made dimensionless in the conventional manner. Crocco's transformation consists in introducing the new independent variables $\xi = x$ and $\eta = u(x, y)$; the shear $u_y = \varphi(\eta, \xi)$ then satisfies the equation $\eta\varphi_\xi - p'(\xi)\varphi_\eta = \varphi^2\varphi_{\eta\eta}$ with the boundary conditions $\varphi_\eta(0, \xi) - p'(\xi)/\varphi(0, \xi) = v_0(\xi)$ and $\varphi[u_1(\xi), \xi] = 0$, where $v_0(\xi)$ represents the flow through the porous wall and $u_1(\xi)$ the potential-flow speed related to $p(\xi)$, both presumed known. According to Carrier and Lin [Quart. Appl. Math. **6**, 63–68 (1948); these Rev. **9**, 477], $\lim_{\xi\to 0} \varphi(0, \xi) = 0.332\xi^{-\frac{1}{2}}$. The author constructs a solution in the form $\varphi(\eta, \xi) = \sum_0^\infty \varphi_n(\xi)\eta^n$, actually using six terms in his numerical work, with the Carrier-Lin initial condition. After $\varphi(\eta, \xi)$ is known, the profile, boundary-layer thickness, drag, etc., can be calculated with some labor. Finally, the stability of the layer can be investigated; using a criterion of Lin [Quart. Appl. Math. **3**, 117–142, 218–234 (1945); **3**, 277–301 (1946); these Rev. **7**, 225, 226, 346]. As examples, the Blasius and Schlichting cases of zero pressure gradient, with and without suction, are treated. The method is then applied to the flow along the upper surface of an airfoil, under several conditions of suction. It is concluded that it is possible to maintain stability all the way to the trailing edge by proper use of suction. *W. R. Sears.*

***Brun, Edmond, et Vasseur, Marcel. Contribution à l'étude thermique de la couche limite laminaire.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 249–263.

A review of developments in the theory of laminar boundary layers in incompressible fluids with constant viscosity and thermal conductivity. There are considered a general class of pressure distributions with particular emphasis on the thermal boundary layer and associated film coefficient. The method of E. Eckert [VDI-Forschungsh. **416** (1942); these Rev. **8**, 236] of referring more general distributions to local conditions on a wedge is extended. The author seems to be unaware, however, of a similar treatment of this theory by Tifford [J. Aeronaut. Sci. **12**, 241–251 (1945)].
N. A. Hall (Minneapolis, Minn.).

Švec, M. E. Heat transmission in the laminar boundary layer on a solid of rotation. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 102–104 (1950). (Russian)

Earlier results of the author [same journal **13**, 257–266 (1949); these Rev. **11**, 277] are extended to obtain polynomial expressions for the velocity and thermal boundary-layer gradients for laminar incompressible flow. The same method of successive approximation is used and the results assume a general external velocity and temperature gradient. The somewhat more specialized data of Fage and Falkner [Aeronaut. Res. Comm., Rep. and Memoranda no. **1408** (1931)] are confirmed and consistency with the data of Pohlhausen [Z. Angew. Math. Mech. **1**, 28–42 (1921)] for a plane is established. *N. A. Hall.*

Schuh, H. The solution of the laminar-boundary-layer equation for the flat plate for velocity and temperature fields for variable physical properties and for the diffusion field at high concentration. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1275**, 19 pp. (1950).

Translated from Zentrale für wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters (ZWB), Berlin-Adlershof. Forschungsbericht Nr. **1980** (1944).

Weber, C. Zur hydrodynamischen Schmiertheorie des Zapfenlagers. Z. Angew. Math. Mech. **30**, 112–120 (1950). (German. English, French, and Russian summaries)

The pressure for a pivot bearing is found approximately by minimizing an integral with an arbitrary added assumption that the pressure is the product of two functions each of one variable. *P. Franklin* (Cambridge, Mass.).

Belyakova, V. K. Concerning the stability of the motion of a viscous fluid in a straight circular tube. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 105–110 (1950). (Russian)

Ce problème a déjà été étudié par Sexl [Ann. Physik (4) **84**(**389**), 807–822 (1927)] en appliquant la méthode des petites oscillations; il trouve que la solution est stable pour des perturbations aux amplitudes suffisamment petites; mais il obtient la solution en supposant des conditions aux limites qui ne sont pas remplies dans la réalité. L'auteur reprend le problème en se posant les conditions aux limites exactes. Elle obtient pour des nombres de Reynolds suffisamment grands une solution approchée qui permet de déterminer le domaine de la stabilité du mouvement.
M. Kiveliovitch (Paris).

***Agostini, L., et Bass, J. Les théories de la turbulence.** Publ. Sci. Tech. Ministère de l'Air, Paris, no. 237, viii+118 pp. (1950).

This book provides a unified exposition of the statistical theory of turbulence as it has been developed in recent years, starting with the work of G. I. Taylor and ending about the middle of 1949. It begins with a chapter on random functions and their use in describing turbulent motion. Chapter 2 describes the various correlation and spectral functions, the special forms which they assume when assumptions such as homogeneity, isotropy and incompressibility are made, and the relations between the different functions. In chapter 3 the implications of the Navier-Stokes equations are discussed. Various similarity hypotheses are treated in a unified fashion. The assumptions underlying Heisenberg's work are discussed and his fundamental equations are derived. In chapter 4 Kolmogorov's similarity theory for locally isotropic turbulence and its relation to the Weizsäcker-Heisenberg theory are discussed. The last chapter takes up the problem of the decay of turbulence behind a grid, in particular, the work of Batchelor and Townsend on the initial and final phases of decay. The work concludes with a list of the fundamental definitions and formulas, some experimental curves obtained by Favre, and a short but adequate bibliography (although the papers

of Loĭcyanskiĭ and Obuhov are missing). In general, the book gives a rather clear summary of the present situation in this field. *J. V. Wehausen* (Providence, R. I.).

Burgers, J. M. Correlation problems in a one-dimensional model of turbulence. I. Nederl. Akad. Wetensch., Proc. 53, 247–260 (1950).

The author studies the equation

$$\partial v/\partial t + v\partial v/\partial y = \nu\partial^2 v/\partial y^2 \tag{1}$$

in order to gain insight into the problem of turbulence. Developments, along the lines of the correlation theory, analogous to the usual ones in the three-dimensional case, are carried out. To obtain additional information, certain exact particular solutions of (1) are obtained and investigated. These solutions demonstrate clearly the increase in the gradient of velocity, the limitation of this process by the effect of viscosity, and the coalescence of the sharp fronts thus formed. The author then proceeds to study correlation functions corresponding to these solutions. [Reviewer's remark. The theory developed in this paper is of course not intended to deal with another aspect of the phenomenon of turbulence, namely, the stretching of vortex filaments. On the other hand, the results obtained here may also be used to demonstrate several aspects of the phenomena of shock formation.] *C. C. Lin* (Cambridge, Mass.).

Kiebel, I. A. Exact solutions of equations of gas dynamics. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1260, 12 pp. (1950).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 193–198 (1947); these Rev. 9, 541.

Wang, Chi-Teh, and Rao, G. V. R. A study of the nonlinear characteristics of compressible flow equations by means of variational methods. J. Aeronaut. Sci. 17, 343–348 (1950).

For a steady, irrotational, isentropic compressible flow Bateman has shown that the continuity equation is the Euler equation for the variational problem $\delta\int p d\tau = 0$, where p is the pressure expressed as a function of the velocity potential. In this paper the author expresses the velocity potential as a sum of two parts. The first of these is that given by the incompressible theory and the second is a series. The coefficients of this series satisfy nonlinear equations obtained from the variational principle. The method is applied to a circular cylinder and indications are obtained to the effect that uniqueness and existence of solutions are in doubt at high Mach numbers. The numerical solutions obtained from using the complete expression for p are compared to those obtained from a linearized expression for p. Good agreement is found in the cases of flow past a circular cylinder, a Kaplan's bump, an elliptical cylinder, and a sphere. *A. H. Taub* (Urbana, Ill.).

Laitone, E. V. The surface pressure coefficient for approximate compressible flow solutions. J. Aeronaut. Sci. 17, 381–383 (1950).

In the nth order Rayleigh-Janzen method for calculating subsonic plane compressible flow, the velocity potential and speed of flow V are approximated by the first $n+1$ terms of power series in the square of the free-stream Mach number, M_∞. Numerical computations indicate that the pressure coefficient C_p should also be approximated only to terms of the same order in M_∞^2 as V, rather than by the usual substitution of V in the exact equation for C_p. Contrariwise, it is asserted that in the first order Prandtl-Glauert method for axisymmetric flow too few terms of C_p are usually retained. *J. H. Giese* (Havre de Grace, Md.).

Schäfer, Manfred, and Tollmien, W. Two-dimensional potential flows. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1243, 24 pp. (1949).

[Translation of "Ebene Potentialströmungen," Technische Hochschule Dresden, Archiv Nr. 44/3, 1941.] The characteristic differential equations for plane isentropic steady potential flow are expressed in terms of the hodograph coordinates. The integration was performed by Meyer [VDI-Forschungsh. no. 62, 31–67 (1908)]. The use of Meyer's function to construct numerical solutions in the manner of Walchner and of Prandtl-Busemann is shown, with examples. Tables are provided. *W. R. Sears.*

Schäfer, Manfred, and Tollmien, W. Rotationally symmetric potential flows. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1244, 31 pp. (1949).

[Translation of "Rotationssymmetrische Potentialströmungen," Technische Hochschule Dresden, Archiv Nr. 44/4, 1940.] The numerical construction of solutions for axisymmetric isentropic flow using the characteristic equations is developed in detail. In this case the differential equation for the characteristics in the velocity plane includes a term proportional to dr, where r is the radial coordinate. This term becomes indeterminate when the characteristic surface is parallel to the axis; the authors avoid this difficulty by introducing the arc length as a coordinate. The method is applied to two practical examples, a diffuser and a nozzle. Tables used in these calculations are included.

W. R. Sears (Ithaca, N. Y.).

Coburn, N. "Characteristic directions" in three-dimensional supersonic flows. Proc. Amer. Math. Soc. 1, 241–245 (1950).

This is a negative paper, designed to show that the author could not have proceeded more simply than he did in a previous paper [the author and Dolph, Proc. Symposia Appl. Math., Vol. I, pp. 55–66, American Mathematical Society, New York, 1949; these Rev. 10, 751].

M. J. Lighthill (Manchester).

Thomas, T. Y. Distribution of pressure on curved profiles in supersonic gas flow with variable entropy. Proc. Nat. Acad. Sci. U. S. A. 36, 109–115 (1950).

In an earlier note, the author presented and computed a first approximation for the pressure along a curved profile in a two-dimensional supersonic flow with attached shock wave. The theory rests on the expression for the pressure in the flow field as a function $p(\omega, S)$ of the flow inclination ω and the entropy S. Assume it possible to expand $p(\omega, S)$ in a series in S about points on (i.e., just aft of) the shock wave. The resulting series allows in principle the calculation of pressures on the profile, since ω is known along the profile and S is constant on it. The first term in the expansion gives the first approximation referred to above. In this paper the author obtains the second approximation by showing that the derivative $\partial p/\partial S$ at points of the shock depends only on ω, the flow inclination behind the shock. The proof rests on a relation to be derived in a forthcoming paper. In view of this result the second approximation is calculable from closed formulas, as was the first. Computation for the case of a biconvex

circular arc profile shows the difference between the first and second approximations to be small over most of the profile. *D. P. Ling* (Murray Hill, N. J.).

Frankl, F. I. On the formation of shock waves in subsonic flows with local supersonic velocities. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1251**, 8 pp. (1950).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 199–202 (1947).

Robbertse, W. P., and Burgers, J. M. Solutions of the equations for the nonuniform propagation of a very strong shock wave. I. Nederl. Akad. Wetensch., Proc. **52**, 958–965 (1949).

A one-dimensional gas flow is characterized by the equation $x=\varphi(s,t)$, where x and s are the Eulerian and Lagrangian coordinates respectively. A shock of very high intensity passes through the gas, with the state of motion of the gas anterior to its passage supposed known. By applying the Lagrangian labels in a special manner a differential relation is found which holds along the trace of the shock. This relation permits the explicit solution of the flow equations provided a particular form for this solution is assumed. In the course of this solution only partial recourse is had to the equation of motion. In order that this equation be fully satisfied it is necessary to postulate a specific density distribution in the unshocked gas. This distribution in turn dictates a shock velocity which increases with time. *D. P. Ling* (Murray Hill, N. J.).

Robbertse, W. P., and Burgers, J. M. Solutions of the equations for the nonuniform propagation of a very strong shock wave. II. Nederl. Akad. Wetensch., Proc. **52**, 1067–1074 (1949). (English. Esperanto summary)

Continuing previous work [see the preceding review], the authors examine numerically the solution already derived. The arbitrary constants appearing in this solution are related to the physical processes involved and evaluated. Further, a numerical comparison is made between this solution and previous approximate solutions. Despite differences in the boundary conditions among the various cases compared, only relatively minor differences appeared in the early phases of the process. *D. P. Ling.*

Wecken, Franz. Grenzlagen gegabelter Verdichtungsstösse. Z. Angew. Math. Mech. **29**, 147–155 (1949). (German. English, French, and Russian summaries)

The author is concerned with stationary triple shock intersections. All the solutions possible according to the shock-wave equations are given by points on an algebraic surface in a M_1^{-1}, ξ_a, ξ_b space, where M_1 is the upstream Mach number and ξ_a, ξ_b are pressure ratios across the two consecutive shocks. He discusses the limiting cases $\xi_a=1$, $\xi_b=1$, $\xi_a=0$, and $\xi_b=0$ at some length, and also the eight "corner points" which represent doubly degenerate limiting cases. Some earlier papers are extended and corrected. *W. R. Sears* (Ithaca, N. Y.).

Munk, M. M. The reversal theorem of linearized supersonic airfoil theory. J. Appl. Phys. **21**, 159–161 (1950).

Under certain conditions, as yet not completely clarified, a wing in supersonic flow has the same drag and lift in the original flow as in the reversed flow according to linearized theory. The author gives a proof, based on simple symmetry arguments of this reversibility theorem. The class of wings for which this proof is valid is not too clearly defined. However, it is specifically assumed that the pressure is computed from the linearized Bernoulli's law so that no leading-edge suction is considered. For the case of zero thickness (lifting case) this condition is satisfied if all edges are supersonic. In this case the angle of attack has to be constant, and, in fact, it is very easy to give counterexamples showing that the lift is changed by flow reversal if the angle of attack is variable. The leading-edge suction is also zero when the average of the local angle of attack on top and bottom surfaces is zero. The local lift is then zero, and the reversibility theorem holds in addition for the drag. *P. A. Lagerstrom* (Pasadena, Calif.).

Harmon, Sidney M. Theoretical relations between the stability derivatives of a wing in direct and in reverse supersonic flow. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1943**, 47 pp. (1949).

W. D. Hayes [thesis, California Institute of Technology, 1947] proved a reversibility theorem for the drag of non-lifting wings in supersonic flow. Several authors (including Hayes) have since proved various reversibility theorems for drag, lift and moments of lifting wings. In the present report an investigation is made by means of the linearized theory of the relations between the supersonic stability derivatives of a wing and the corresponding derivatives of the same wing when the flow direction is reversed. The analysis includes wings that have a supersonic leading edge, a supersonic trailing edge, and either supersonic side edges or non-reentrant subsonic side edges which are noninteracting, that is, neither edge lies in the zone of action of the other. The investigation is made for thin flat-plate wings and includes steady vertical and longitudinal motions, steady rolling and pitching, and motions in which the wing is accelerating uniformly in the vertical direction.

The results of the investigation are obtained with respect to body axes and indicate that the rates of change of lift and drag with angle of attack and forward speed, the damping in roll, and the rate of change of lift with a uniform variation of angle of attack with time are all equal for the wing and its reverse. Also, with respect to axes of rotation that are fixed with respect to the wing configuration, the damping in pitch is equal for the wing and its reverse; whereas the rate of change of pitching moment with angle of attack for the wing is equal to the rate of change of lift with pitching velocity multiplied by the stream velocity for the reverse of the wing. [Cf. the following review.] *P. Lagerstrom* (Pasadena, Calif.).

Brown, C. E. The reversibility theorem for thin airfoils in subsonic and supersonic flow. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1944**, 9 pp. (1949).

The author has improved an idea of M. Munk and obtained general reversed-flow theorems for lift and related quantities in a linearized fluid flow. The main principles used are that of superposition and that of the representation of drag in the wave and vortex systems, and the theorems hold for very general bodies in either subsonic or supersonic flow. The Kutta condition is essential to the validity of the theorems. The new theorems proved are that for any body satisfying planar system or similar restrictions the lift curve slope $dC_L/d\alpha$, the damping in roll C_{l_p}, and the damping in pitch C_{m_q} are unchanged by a reversal of the flow direction.

The author points out an error in a paper of W. Hayes [Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 412–424; these Rev. **11**, 554], who had mistakenly concluded that this lift theorem could not be general. Other

authors, for example G. N. Ward, A. Flax, and S. Harmon [cf. the preceding review] have given similar theorems but with much less generality. *W. D. Hayes.*

Ward, G. N. Supersonic flow past thin wings. II. Flow-reversal theorems. Quart. J. Mech. Appl. Math. **2**, 374–384 (1949).

Two theorems are proved. (I) The equality of drag and side force for a symmetrical thin wing and the same wing in reversed flow, (II) the equality of lift-curve slopes for a thin wing in forward and reversed flow, provided that the flows at its subsonic edges (if any) are independent and that the Kutta-Joukowski condition holds at subsonic trailing edges. Theorem (I) is easily proved from formulas of an earlier paper [same vol., 136–152 (1949); these Rev. **11**, 64]. Theorem (II) requires separate consideration of various contributions to the lift, and the proof is more difficult. After proving theorem (II) the author digresses slightly to obtain a very simple formula for the lift of a wing having no subsonic edges and whose trailing edge is straight. Finally, the possibility of a more general lift theorem than (II) is discussed. In particular, it is shown by means of the simple lift formula just derived that a lift-reversal theorem cannot hold in general. Reference is made to the work of Hayes [Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 412–424; these Rev. **11**, 554]. [For additional recent work on this subject see Flax, J. Aeronaut. Sci. **16**, 496–504 (1949); these Rev. **11**, 224; see also the preceding review.] *W. R. Sears* (Ithaca, N. Y.).

Heaslet, Max. A., and Lomax, Harvard. Two-dimensional unsteady lift problems in supersonic flight. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **945**, 9 pp. (1949).

Formerly issued as Tech. Notes Nat. Adv. Comm. Aeronaut. no. **1621** (1948); these Rev. **9**, 478.

Haskind, M. D., and Falkovich, S. V. Vibration of a wing of finite span in a supersonic flow. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1257**, 11 pp. (1950).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 371–376 (1947); these Rev. **9**, 114.

Harmon, Sidney M., and Jeffreys, Isabella. Theoretical lift and damping in roll of thin wings with arbitrary sweep and taper at supersonic speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2114**, 49 pp. (1950).

Thomas, Richard N. Supersonic flow past a cone and wedge. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. **483**, 11 pp. (1944).

Author's summary. By a change of the variables expressing the shock wave conditions, somewhat simpler formulae for computing the flow past a wedge and cone are obtained. In particular, the methods yield explicit values for the limiting (high velocity) case for the cone as well as the wedge. *W. R. Sears* (Ithaca, N. Y.).

★Hirsch, René. Détermination et calcul des hélices d'avions optima, simples et coaxiales. Tome I. Publ. Sci. Tech. Ministère de l'Air, Paris, no. **220**, viii+192 pp. (11 plates) (1948).

★Hirsch, René. Détermination et calcul des hélices d'avions optima, simples et coaxiales. Tome II. Publ. Sci. Tech. Ministère de l'Air, Paris, no. **225**, ii+129 pp. (1949).

This is an exhaustive treatise on the theory of propellers, begun before 1939. Its features are the attempts to account for compressibility effects, the treatment of dual-rotating coaxial propellers, and the consideration of structural and aerodynamic factors together in optimum-propeller design. Reference is made in a number of instances to German results, especially in connection with compressibility effects on airfoils. In calculating the induced velocities, the author employs an approximate system made up of discrete helical trailing vortices. For dual-rotating propellers he estimates the periodic variation of blade circulation, including a rough account of unsteady-lift effects, and decides that it can be neglected. Supposing that the optimum circulation distribution has been determined, the author proposes to design the propeller blades so as to minimize the efficiency loss due to drag. The drag increases torque and reduces thrust. The problem becomes one of optimizing the function lC_x, where l is the blade chord and C_x the local drag coefficient, which depends on skin friction, profile curvature, lift coefficient, thickness ratio, Reynolds number, and Mach number, and at the same time meeting the requirements of structural strength. The last two chapters are devoted to the problems of propeller vibration and the dynamics of propellers with automatic blade-angle regulation. In these cases unsteady-lift effects are neglected. *W. R. Sears* (Ithaca, N. Y.).

Bechert, C. Sur la théorie de la propagation des combustions, avec une application à la combustion de l'ozone. Portugaliae Phys. **3**, 29–90 (1949).

The theoretical determination of normal flame velocity u_l from the basic principles of hydrodynamics and chemical kinetics was first made by Lewis and von Elbe [J. Chem. Phys. **2**, 537–546 (1934)] for the ozone decomposition reaction. To make the computation tractable, these authors were forced to use an important simplifying assumption. A complete but very complex formulation of this problem was given by Hirschfelder and Curtiss [J. Chem. Phys. **17**, 1076–1081 (1949)]. The present paper departs from both these treatments in that no attempt is made to give a fundamental theory; only semi-empirical or phenomenological analysis is given. This is achieved through the introduction of the concept of a single "overall reaction" with an activation energy E. The molar fraction of the combustible is denoted by $1/(1+m^*)$, the ratio of molecular weights of the combustible and the product by μ. Then the result of calculation shows that

$$(1)\qquad u_l=\frac{(2A_1)^{\frac{1}{2}}(m^*+1)(m^*+\mu)^{-\frac{1}{2}}e^{-\tau_r/2}}{\tau_l\tau_r(1-\tau_r/\tau_l)^{\frac{1}{2}}},$$

where the τ's are defined as $\tau\equiv E/kT$, k the Boltzmann constant, and the subscripts l and r refer to the unburned and the burned gas; A_1 is related to the properties of the reacting molecules. The author considers A_1 and E as constants to be determined from, say, two experimental measurements on u_l. Then (1) can be used to predict u_l under other conditions. The author obtains good correlation with experiments on the decomposition of ozone by Lewis and von Elbe [J. Chem. Phys. **2**, 283–290 (1934)].

For the detailed calculation, the author uses three equations, the continuity, the momentum and the energy equation. The viscous stress is neglected in the momentum equation and the diffusion is accounted for by an appropriate modification of E. Introducing the boundary condition that $(d^2T/dx^2)_l=0$, x space coordinate, the author shows that $(dT/dx)_l$ is very small, i.e., the space variations of temperature and other physical variables in the unburned gas are

very small. The equations are then reduced to a single first order differential equation with the independent variable τ, and the parameter u_l. Since there are two end conditions at the unburned side and the burned side to be satisfied, the parameter u_l can be determined. The actual solution is done by first assuming a proper approximate solution which satisfies the end conditions and then "forcing" this solution to fit the differential equation at some point. The choice of the fitting point does not influence the result greatly and (1) is the averaged result. *H. S. Tsien* (Pasadena, Calif.).

Martin, M. H., and Jackson, G. B. The sound waves generated by a particle at supersonic speed. Schweiz. Arch. Angew. Wiss. Tech. 16, 114–119 (1950).

The authors determine the acoustic radiation pattern from a particle moving under the action of a constant gravitational field and a resisting force depending only on the velocity and having a subsonic limiting speed. It is shown that the family of waves simply covers all space, that a fixed observer will hear the particle only once, and that the apparent source is uniquely determined. *J. W. Miles.*

Harkevič, A. A. Power horns and Stokes polynomials. Doklady Akad. Nauk SSSR (N.S.) 68, 685–688 (1949). (Russian)

An acoustic horn is discussed which has cross-sectional area varying as x^{2n} where x is distance along the axis of the horn. The pressure p in the horn is given by an expression involving Bessel functions of $\omega x/c$ which reduces, for integer n, to the form $p=F(kx)\exp(-ikx)$. Here $k=\omega/c$ and F is a polynomial in $1/kx$. The characteristic impedance of the horn is then a rational function of ω and has an equivalent circuit of ladder type. *E. N. Gilbert.*

Harkevič, A. A. A new method for solving diffraction problems. Doklady Akad. Nauk SSSR (N.S.) 72, 45–47 (1950). (Russian)

The method is to reduce the problem to a solved diffraction problem by a coordinate transformation. This is achieved in the case of the motion of a semi-infinite plane shock-wave (zero pressure before the wave-front, unit pressure at all points behind it) over a rigid half-plane screen, diffraction starting when the shock-front reaches the edge. This is reduced to the problem of a similar wave radiated normally away from the screen. The author thus confirms his previous work on the original problem [Akad. Nauk SSSR. Žurnal Teh. Fiz. 19, 828–832 (1949)]. *F. V. Atkinson.*

Bouwkamp, C. J. On the freely vibrating circular disk and the diffraction by circular disks and apertures. Physica 16, 1–16 (1950).

The author summarizes his paper as follows. "The author develops a theory of the acoustic field produced by a freely vibrating, rigid, circular disc on the assumption that the wave length is large compared to the radius of the disc. The solution is presented in the form of a series of ascending powers of wave number times the radius of the disc. The new approach, which is based on integral equations, easily permits the explicit calculation of a number of terms of this series. The results are equally applicable to the diffraction of circular discs and apertures by plane waves impinging in the normal direction upon an obstacle. A summary of earlier results by various authors is included."

A. Heins (Pittsburgh, Pa.).

Storruste, A., and Wergeland, H. On two complementary diffraction problems. I. Circular hole and disc in confocal coordinates. Norske Vid. Selsk. Forh., Trondheim 21, no. 10, 38–42 (1948).

Storruste, A., and Wergeland, H. On two complementary diffraction problems. II. Transmission of sound through a circular hole. Norske Vid. Selsk. Forh., Trondheim 21, no. 11, 43–48 (1948).

The authors are concerned with the diffraction of a normally incident plane wave of sound by a circular disk or hole in an infinite plane screen. Both the disk and the screen are assumed to be rigid and infinitely thin. The wave equation is separated in oblate-spheroidal coordinates and the wave field expanded in spheroidal wave functions of order zero. A simple expression is derived for the transmission coefficient of the circular hole. [The authors would like to make the following corrections: (1) the paragraph relating to Casimir should be deleted (p. 39); (2) the M-functions are finite at $\mu=0$ (p. 45, lines 5 and 6 from bottom); (3) in the addendum (p. 47, line 3 from bottom) "these authors" refers to Levine and Schwinger.] *C. J. Bouwkamp.*

Storruste, A. Transmission of waves through circular apertures. Norske Vid. Selsk. Forh., Trondheim 21, no. 21, 84–87 (1949).

Storruste, A. Scattering of waves by circular discs. Norske Vid. Selsk. Forh., Trondheim 21, no. 22, 88–91 (1949).

Numerical evaluation of the solution obtained by Storruste and Wergeland [see the preceding review] when the circumference of the disk or hole is ten wavelengths or less. Curves are given for the scattering (transmission) coefficient of the disk (hole) for the case of normal incidence.

C. J. Bouwkamp (Eindhoven).

Ursell, F. On the theoretical form of ocean swell on a rotating earth. Monthly Notices Roy. Astr. Soc. Geophys. Suppl. 6, 1–8 (1950).

In the discussion of ocean swell the waves studied by Gerstner [1802] and Stokes [1847] have received particular attention. One of the fundamental assumptions underlying these theories is that the wave motion occurs on a nonrotating earth. The author shows that if the curvature of the earth can be neglected, the effect of the earth's rotation makes swell waves in their stationary state differ very little from Gerstner waves. The work suggests that in the general case each particle moves approximately in a horizontal circle of inertia as well as in the nearly vertical Gerstner motion. The diameter of the inertia circles is greatest at the surface, where it may be several hundred metres, and the period of revolution is about 12 csc l hours where l is the latitude. Except for this circular movement there is no mass transport, and thus there is no ground for supposing that ocean drift currents are due to fluid transport with the waves. On the other hand, waves appear to provide a reasonable mechanism for the generation of inertia currents which have been frequently observed. *L. M. Milne-Thomson.*

Rees, M. R. The equilibrium distribution of the long-period tides over an ocean covering the northern hemisphere. Quart. J. Mech. Appl. Math. 3, 80–88 (1950).

The author gives the results of a calculation of the equilibrium distribution if the long-period tides over an ocean covering the northern hemisphere of a uniform spherical globe, making allowance for the change in the gravitational

field produced by the tides themselves. The result is to give an elevation which very closely approximates that which would be obtained by multiplying the uncorrected level by 1.126. *L. M. Milne-Thomson* (Greenwich).

✱Wilkes, M. V. Oscillations of the Earth's Atmosphere. Cambridge University Press, 1949. x+76 pp. $2.50.

The ocean tides can be studied from variations of depth measurement. Atmospheric tides can be inferred only from variations of barometric pressure. Laplace applied his ocean tide theory to infer that in the tropics a lunar atmospheric tide of amplitude about 0.25 mm. should exist. The first reliable determination of the lunar atmospheric tide is due to Lefroy, director of the observatory at St. Helena, who found [1842] an amplitude of 0.06 mm. In 1918 Chapman established the existence at Greenwich of a semi-diurnal lunar tide of amplitude about 0.01 mm. of mercury. The present volume discusses all that is known to date of the lunar and solar air tides. It appears that their general features are well explained but that theory throws but little light on such aspects as seasonal and annual changes. The resonance theory may now be taken as well established. The great regularity, symmetry and large amplitude of the solar semi-diurnal barometric variation are strong arguments in its favour and the results of rocket experiments and other evidence have now shown that the temperature of the atmosphere varies with height in the right kind of way to give a free solar period of 12 hours. The contents are as follows. (I) The lunar and solar airtides. Historical introduction, solar barometric variation, lunar barometric variation, lunar tides in the ionosphere. (II) The theory of oscillation in a rotating atmosphere. General equations, forced oscillations, boundary conditions, general solution, free periods, propagation of a pulse, atmosphere in adiabatic equilibrium, damping. (III) Numerical evaluation of airtides. Laplace's tidal equation, evaluation. (IV) Oscillations in the atmosphere. The outward flux of energy, temperature variation in the atmosphere, oscillations of the earth's atmosphere (general treatment and numerical results). (V) Discussion of results. *L. M. Milne-Thomson.*

Elasticity, Plasticity

✱Filonenko-Borodič, M. M. Teoriya uprugosti. [Theory of Elasticity]. 3d ed. OGIZ, Moscow-Leningrad, 1947. 300 pp.

A considerably revised and rewritten version of a book originally issued in 1932 [2d ed., 1933] and intended as an introduction to elasticity theory for students in technical schools and universities. Chapter headings: I. Theory of stress. II. Geometrical theory of strain. III. The generalized Hooke's law. IV. Solution of the displacement problem of the theory of elasticity. V. Solution of the stress problem of the theory of elasticity. VI. The plane problem in Cartesian coordinates. VII. The plane problem in polar coordinates. VIII. Torsion and bending of prismatic rods. IX. More general methods of solution of the problems of the theory of elasticity. X. The bending of flat plates.

Signorini, A. Trasformazioni termoelastiche finite. II. Ann. Mat. Pura Appl. (4) 30, 1–72 (1949).

This paper is the latter half of a systematic exposition of the author's researches in the theory of finite elastic strain [for part I see the same Ann. (4) 22, 33–143 (1943); these Rev. 8, 708]. Chapter I contains an amplification of his earlier treatment of the uniqueness (and, in certain cases, nonexistence) of solutions of the general equations when all quantities, including the loads, are expanded as formal power series in an arbitrary parameter. Chapter II deals with the author's theory of second degree elasticity, in which the stresses are only quadratic functions of the finite strain components and yet are exactly derivable from a strain energy. Most of the main results were obtained previously by the author and by Tolotti in papers which are cited in the present work. *C. Truesdell* (Bloomington, Ind.).

✱Štaerman, I. Ya. Kontaktnaya zadača teorii uprugosti. [The Contact Problem of the Theory of Elasticity]. Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoĭ Literatury, Moscow-Leningrad, 1949. 270 pp.

Roma, Maria Sofia. Integrazione del sistema di equazioni dell'elastostatica tridimensionale in un manicotto cilindrico illimitato. Ann. Scuola Norm. Super. Pisa (3) 2 (1948), 63–83 (1950).

The author studies the deformation of a three-dimensional elastic body, body forces being absent, which occupies the space between two coaxial right circular cylinders. Let Σ' and Σ'' be the inner and outer cylinders, respectively. Three boundary value problems are solved in the first section: (1) displacement prescribed on Σ' and Σ'', (2) normal stress prescribed on Σ' and Σ'', and (3) displacement prescribed on Σ' and normal stress prescribed on Σ''. The second section contains the determination of the displacement when there is only one cylinder present and it is subjected to axially symmetric pressure. The Fourier transform method is employed in the solution of all problems considered.

J. B. Diaz (College Park, Md.).

Flügge, W., und Marguerre, K. Wölbkräfte in dünnwandigen Profilstäben. Ing.-Arch. 18, 23–38 (1950).

This paper is a general analytic discussion of the torsional warping-restraint problem in thin-walled, uniform cross-section rods of both open and closed construction. Particular emphasis is placed on pointing out the common basis, although very different character, of the mechanism in the two types of structure. Rods with **Z** section (flanges and web perpendicular) and symmetrical tubes of rectangular cross-section, both with and without corner stringers, are dealt with in detail as examples. *M. Goland.*

Levi, Franco. Superfici d'influenza e fenomeni di adattamento nelle lastre piane. Ricerca Sci. 20, 482–486 (1950).

Burr, A. H. Longitudinal and torsional impact in a uniform bar with a rigid body at one end. J. Appl. Mech. 17, 209–217 (1950).

Freiberger, W. The uniform torsion of an incomplete tore. Australian J. Sci. Research. Ser. A. 2, 354–375 (1949).

This problem is of practical interest in connection with the calculation of stresses in close-coiled helical springs. The author determines a stress system which is equivalent to a single external force acting along the axis of the tore. The problem reduces to the determination of a single stress function which can be expressed in toroidal coordinates and satisfy the boundary conditions on the free surface of the tore. The stresses and displacements are evaluated in terms

of the dimensions of the tore and compared with the results of known approximate theories. *D. L. Holl.*

Storchi, Edoardo. Sulle equazioni indefinite della statica delle membrane tese su generiche superficie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 116–120 (1950).

This paper deals with the equilibrium of membranes, which are regarded as two-dimensional Riemannian geometries over which a stress system is defined but no stress-strain relation is assumed. By employing isothermal coordinates and introducing a transformation, the author reduces the equilibrium equations to the form

$$(*)\qquad \begin{aligned}\partial A/\partial x+\partial B/\partial y&=(A+C)\partial\mu/\partial x,\\ \partial B/\partial x+\partial C/\partial y&=(A+C)\partial\mu/\partial y,\end{aligned}$$

where A, B, C are proportional to the stress components and μ is a given function. For the case when the first stress invariant vanishes ($A+C=0$) he obtains the general solution of (*) in a form closely analogous to Airy's solution for the plane. Then he tries to solve the system (*) in general by putting A, B, C equal to linear combinations of a single scalar and its first and second partial derivatives. He concludes that such a representation leads to a solution if and only if the membrane is of constant Gaussian curvature. The resulting solution is that obtained by Finzi [same Rend. Cl. Sci. Fis. Mat. Nat. (6) 19, 578–584, 620–623 (1934); cf. also the author, ibid. (8) 7 (1949), 227–231 (1950); these Rev. 11, 556]. *C. Truesdell* (Bloomington, Ind.).

Wyman, Max. Deflections of an infinite plate. Canadian J. Research. Sect. A. 28, 293–302 (1950).

In order to ascertain the load-carrying capacity of floating ice sheets, the author investigates the small-deflection problem of the bending of an infinite isotropic plate on an elastic foundation. The general solution, in terms of modified Bessel functions, is used to discuss the case of (a) a concentrated normal load, and (b) uniform loading over a circular area. To obtain the results for the second case from those of the first, the addition theorem for kei x is found, as also are two identities between the functions ber x, bei x, ker x, kei x, and their derivatives. *H. D. Conway.*

Richard, Ubaldo. Sul problema della piastra incastrata. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 83, 21–27 (1949).

With the aid of an appropriate system of orthogonal harmonic functions the solutions of the static and dynamic problem of the thin elastic plate clamped at the edge are expressed in terms of a new expression for the Green function of the biharmonic equation. *P. E. Neményi.*

Chang, Fo-Van. Trigonometric series applied to the bending of long rectangular plates to a cylindrical surface. J. Franklin Inst. 249, 279–286 (1950).

Belluzzi, Odone. Lo studio delle strutture costituite da lastre curve. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 5 (1947/48), 3–9 (1949).

✶Carrier, G. F. On dynamic structural stability. Proc. Symposia Appl. Math., Vol. I, pp. 175–180. American Mathematical Society, New York, N. Y., 1949. $5.25.

The author considers structures for which small displacements correspond to suitably chosen systems of surface tractions of the form $\lambda T_i A(t)$, where T_i denotes a static system, $A(t)$ is a function of time, and λ is a constant parameter. It is further postulated that the stresses and displacements corresponding to this system are proportional to λ. Dynamic instability may then occur for certain ranges of λ and certain functions $A(t)$, in which cases the transient oscillations increase in amplitude with time. The relevant mathematical problem is formulated in terms of a system of equations, and also in terms of a variational principle. Comparison is made with a formulation due to W. Prager [Quart. Appl. Math. 4, 378–384 (1947); these Rev. 9, 120] in the special case of static loading, and an explicit consideration of dynamic instability of a thin circular plate subjected to a uniform radial pressure is included.

F. B. Hildebrand (Cambridge, Mass.).

Szegö, G. On membranes and plates. Proc. Nat. Acad. Sci. U. S. A. 36, 210–216 (1950).

Let D be a plane domain having a smooth simple closed curve C as boundary. Define the positive numbers λ_1, λ_2, λ_3 as follows:

$$\text{(a)}\qquad \lambda_1^2=\min\frac{\iint_D|\operatorname{grad} u|^2d\sigma}{\iint_D u^2d\sigma},\qquad u=0 \text{ on } C,$$

$$\text{(b)}\qquad \lambda_2^4=\min\frac{\iint_D(\nabla^2u)^2d\sigma}{\iint_D u^2d\sigma},\qquad u=\frac{\partial u}{\partial n}=0 \text{ on } C,$$

$$\text{(c)}\qquad \lambda_3^2=\min\frac{\iint_D(\nabla^2u)^2d\sigma}{\iint_D|\operatorname{grad} u|^2d\sigma},\qquad u=\frac{\partial u}{\partial n}=0 \text{ on } C,$$

the identically zero function being excluded. The numbers λ_1 and λ_2 are the fundamental frequencies of a membrane with fixed boundary and of a clamped plate, respectively, and (c) occurs in the study of buckling of plates [see A. Weinstein, Étude des spectres des équations aux dérivées partielles . . . , Mémor. Sci. Math. no. 88, Paris, 1937]. Concerning (a), Rayleigh conjectured that, of all membranes of a given area, the circular one has the lowest fundamental frequency. This conjecture was first proved by G. Faber [S.-B. Math.-Phys. Kl. Bayer. Akad. Wiss. München 1923, 169–172], making use of the known fact that no minimizing function of problem (a) has a zero in D. The author first gives a different proof of Rayleigh's conjecture, still making use of the nonvanishing in D of any minimizing function of problem (a). By modifying this last argument, the author shows that if it is true that in problems (b) and (c) no minimizing function has a zero in D, then the analogue of Rayleigh's conjecture holds; i.e., of all plane domains of a given area, the circular one has the lowest λ_2 and λ_3.

J. B. Diaz (College Park, Md.).

Pignedoli, Antonio. Frequenze di vibrazione di una membrana elastica a contorno epicicloidale fisso. Ann. Mat. Pura Appl. (4) 30, 291–307 (1949).

The author presents a method for approximating the frequencies of vibration of an elastic membrane held fixed along its epicycloidal boundary (supposed free of loops and cusps). The eigenvalue problem consists in deter-

mining the eigenvalues μ^2 of the partial differential equation $\partial^2 u(x, y)/\partial x^2+\partial^2 u(x, y)/\partial y^2+\mu^2 u(x, y)=0$, subject to the boundary condition $u=0$ on the boundary. Use is made of the known conformal transformation of the complex number plane onto itself which maps the interior of (the exterior of) the unit circle onto the interior of (the exterior of) the epicycloid. The investigation leads to the introduction of certain transcendental functions, analogous to Bessel functions, which are termed epicycloidal functions.

J. B. Diaz (College Park, Md.).

Giangreco, Elio. Sulle vibrazioni delle piastre con nervature. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 34–38 (1950).

The author extends the results of Tolotti and Grioli [Giornale del Genio Civile, no. **6** = Pubbl. Ist. Appl. Calcolo no. **258** (1949); these Rev. **11**, 701] and his own [forthcoming] concerning the static deflection w of elastic plates reinforced with ribs to the elastodynamic case. He limits his attention, however, to the problem of a freely vibrating rectangular plate with two central ribs rigidly attached thereto which intersect one another orthogonally. Employing the simplified method developed by Tolotti and Grioli [loc. cit., chapter 2] for the case of plates reinforced by rigidly attached ribs, in conjunction with the principle of d'Alembert, and confining himself to the case when $w=\bar{w}(x, y)\cos\omega t$, the author derives the homogeneous integro-differential equation which governs $\bar{w}$. By developing the kernel therein in terms of its characteristic functions and terminating the series after a finite number of terms, he obtains successive approximations to the natural frequencies of vibration of the plate by standard procedures. A few preliminary results for the fundamental frequency are included.

A. W. Sáenz (Washington, D. C.).

Giangreco, Elio. Sulle vibrazioni delle piastre con nervature. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 113–115 (1950).

The author presents in detail the results announced in an earlier paper [see the preceding review] concerning the fundamental frequency of vibration of a free rectangular plate with two central and orthogonal ribs rigidly attached thereto by employing a well-known result of thin plate theory [A. Nádai, Die elastischen Platten, Springer, Berlin, 1925, p. 174]. He uses his procedure to compute the fundamental frequency of the plate in first and second approximation. A numerical example is included to illustrate the rapid convergence of the method.

A. W. Sáenz (Washington, D. C.).

Ruppeneĭt, K. V. Compression of a cylinder between two rough rigid plates. Doklady Akad. Nauk SSSR (N.S.) **72**, 247–250 (1950). (Russian)

A cylindrical sample is compressed between two rough rigid plates under the assumption that plane sections perpendicular to the generatrix remain plane. The general yield condition is written in the form

$$(\sigma_z-\sigma_r)^2+4\tau_{rz}^2=4k^2\sin^2(\{(\sigma_z+\sigma_r)/2k\}+H/r),$$

where σ_z and σ_r are the axial and the radial stresses, respectively, and k and H are constants. The problem is solved for this condition as well as for a perfectly plastic material. It is pointed out that in the latter case the uniaxial state of stress usually assumed in engineering approximations holds only when the height of the sample h satisfies the relation $h\geqq 2R_0\cot\xi_0$, where R_0 is the radius of the sample and ξ_0 is given by $H=k(2\xi_0-\sin 2\xi_0)$. *H. I. Ansoff.*

Swida, W. Über die Restspannungen bei der elastischplastichen Biegung des krummen Stabes. Ing.-Arch. **18**, 77–83 (1950).

In an earlier paper [Ing.-Arch. **16**, 357–372 (1948); these Rev. **10**, 497] the author discussed the elastic-plastic bending of a curved bar during the initial loading. In the present paper this investigation is extended to include the residual stresses after unloading. *W. Prager* (Providence, R. I.).

Galin, L. A. The elastic-plastic torsion of prismatic bars. Grad. Div. Appl. Math. Brown Univ. Translation **A11-T9**, 22 pp. (1949).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 285–296 (1949); these Rev. **11**, 70.

Thorne, C. J. On plastic flow and vibrations. J. Appl. Mech. **17**, 84–90 (1950).

This paper concerns the mechanical response of a model consisting of two elastic elements and one viscous element, in a series-parallel arrangement, together with a lumped mass. Four cases are treated in detail. In particular, the steady state response to periodic excitation is given in terms of material constants for any frequency. The cases of constant force, constantly increasing force, and constantly increasing elongation are also treated. *T. Alfrey, Jr.*

Kočetkov, A. M. On the propagation of elastic-viscous-plastic shear waves in a plate. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 203–208 (1950). (Russian)

A circular hole is punched in an infinite thin plate. The punch is assumed rigid and the plate is assumed to deform in shear only, the shear strain being constant across the thickness. The stress-strain expression used beyond the yield limit is of the flow type, the rate of shear strain $\dot{\gamma}_{rz}$ being given by a linear combination of the quantity $\tau_{rz}-F(\gamma_{rz})$ and its derivative, where τ_{rz} is the shear stress and $F(\gamma_{rz})$ is an empirical function. A solution is presented for the ideally plastic case in which $F(\gamma_{rz})=\tau_s$, where τ_s is the yield stress, as well as for the case of linear hardening in which $F(\gamma_{rz})=\tau_s[1+(n^2-1)(\gamma_{rz}/\gamma_s-1)]$, where γ_s is the yield strain and $1<n<2^{1/2}$. Graphs are plotted for the ideally plastic case showing stress and velocity histories in the plate as well as the distribution of the residual strain. *H. I. Ansoff.*

***Goluškevič, S. S. Ploskaya zadača teorii predel'nogo ravnovesiya sypučeĭ sredy. [The Plane Problem of the Theory of the Limiting Equilibrium of Granular Substances].** OGIZ, Leningrad-Moscow, 1948. 148 pp. (4 plates).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Steel, W. H. The distribution of light from optical systems. Australian J. Sci. Research. Ser. A. 2, 335–353 (3 plates) (1949).

The author gives formulae for the light distribution in a plane perpendicular to the axis for a point light source on the axis, and approximate formulae for a point off the axis. He discusses modifications due to the finite, but small size of the light source. Since the values so obtained become infinite near the caustic, he introduces a special definition to compare the brightness of the points of the caustic [cf. W. R. Hamilton, Collected Works, vol. I, pp. 51 ff.]. The results are compared with experiment, by using a reflector button as an optical system. *M. Herzberger* (Rochester, N. Y.).

Speiser, Andreas. Il gruppo metrico dei colori. Ann. Mat. Pura Appl. (4) 28, 231–236 (1949).

Using the group-theoretic development of elementary geometry [G. Thomsen, Grundlagen der Elementargeometrie, Teubner, Leipzig-Berlin, 1933], the author shows that the Euclidean character of color-space can be based solely upon the notion of the identity of two colors. Its metric he calls intrinsic or objective. Contrasted with this is the subjective metric which must be based upon a law of discrimination such as Fechner's. Provisionally he derives for this a form $(2t-t^2)z^{-2}(dx^2+dy^2)+16\pi^{-2}(1+z^2)^{-2}dz^2$, $t=(4/\pi)\tan^{-1}z$. *A. S. Householder* (Oak Ridge, Tenn.).

Fano, R. M. A note on the solution of certain approximation problems in network synthesis. J. Franklin Inst. 249, 189–205 (1950).

The author discusses certain known approximation problems for transmission coefficients of finite linear networks by means of conformal representation of the complex frequency plane $\lambda\rightarrow z$ and the electrostatic potential analogy. In the case of a low pass filter the transformation $\lambda=\sinh z$ can be made to reproduce the pass-band of frequencies $j\omega$ of the λ-plane periodically on the whole imaginary axis of the z-plane. The author stresses the fact that in the plane of the new variable any uniform and symmetric distribution of sources for the potential functions of which the analytic function for the square of the transmission coefficient is formed presents itself to one's intuition as a means of producing the desired Tchebychef behaviour in the pass-band. In order to evaluate transmission coefficients with Tchebychef approximation of a positive constant in the pass-band, and of zero in the rejection band, a transformation by means of elliptic functions is used; another set of uniformly and symmetrically distributed sources generates the required potential functions.

A. Gonzáles Domínguez (Buenos Aires).

Baumann, Ernst. Über Scheinwiderstände mit vorgeschriebenem Verhalten des Phasenwinkels. Z. Angew. Math. Physik 1, 43–52 (1950).

The author determines rational impedance functions whose phase angle approximates a prescribed constant value inside a given band of frequencies and increases monotonically with frequency outside this band. The approximation is of the Tchebychef type. The parameters of the impedance functions are evaluated in terms of elliptic functions by means of a method very similar [as the author points out himself] to that used by Cauer for solving the analogous approximation problem for the modulus of a rational transfer impedance. *A. Gonzáles Domínguez* (Buenos Aires).

Piloty, Hans. Die Brücken-Reaktanzen eines symmetrischen Filters mit vorgeschriebenem Betriebsverhalten. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1947, 129–138 (1949).

The author discusses reactance four-pole networks which produce a desired gain and phase from a constant resistance source at the input to a constant resistance at the output. The networks are symmetric lattices. The main results have already been given by S. Darlington [J. Math. Physics 18, 257–353 (1939); these Rev. 1, 275] but are presented differently here. *E. N. Gilbert* (Murray Hill, N. J.).

Tomiyasu, K. Unbalanced terminations on a shielded-pair line. J. Appl. Phys. 21, 552–556 (1950).

When an unbalanced radiative load terminates a two wire shielded transmission line, two TEM propagating modes will generally exist. The author uses the term "balanced mode" to denote the conventional mode where the currents on the two conductors are equal in magnitude and opposite in phase; whereas the term "unbalanced mode" denotes the condition where the currents on the two conductors are equal in magnitude and equal in phase. The author shows that the characteristics of an unbalanced termination can be obtained from the difference between the standing-wave distributions on the two conductors. From such standing-wave measurements, the reflection coefficients of the termination on each conductor separately are then computed. The balanced and unbalanced reflected modes can then be calculated from these data with the aid of orthogonal plots of certain complex parameters also obtained algebraically from the measured reflection coefficients. A discussion of experimental techniques and the results of some measurements are also included. *R. Kahal.*

Buchholz, Herbert. Berechnung von Wellenwiderstand und Dämpfung von Hochfrequenzleitungen vom Feldbild des vollkommenen Leiters her. I, II. Arch. Elektrotechnik 39, 79–100 (1948).

The first part gives an exposition of the well-known theory of propagation of electromagnetic waves along a pair of cylindrical ideal conductors. This is supplemented by an approximate treatment of the loss due to finite conductivity at very high frequencies on the basis of a penetration depth small compared to the linear dimensions and radii of curvature of the conductor cross-sections as well as of the half wavelength. The two important quantities are the characteristic impedance Z and attenuation constant β. Conformal mapping is applied. The problem is reduced to finding the function $u(s)$ which maps the contours of the conductors onto two parallel sides of a rectangle, and Z and β are expressed in terms of the characteristic parameter contained in $u(s)$ and of an integral over one side of the rectangle involving $|dz/du|^{-1}$, respectively. The general character of the mapping function is shown to be essentially doubly periodic. In the second part, application is made to a variety of concentric line cross-sections, with a strip inside a circular cylinder as prototype. Application of the Schwartz-Christoffel method leads to solution of the associated mapping problem in terms of Jacobian elliptic functions. By

means of the Landen transform, the complementary configuration is covered, too, with application to open-line structures made in the third part. From this prototype, there results a whole pencil of cross-sectional shapes according to the equipotential lines of the doubly slit x-axis. These include cross-sections of practical importance. The quantities Z and β are computed and plotted versus a parameter characteristic of the cross-sectional width-to-height ratio. It turns out that some shapes are somewhat superior to concentric circular cylinders in regard to attenuation.

H. G. Baerwald (Cleveland, Ohio).

Buchholz, Herbert. Berechnung von Wellenwiderstand und Dämpfung von Hochfrequenzleitungen vom Feldbild des vollkommenen Leiters her. III. Arch. Elektrotechnik 39, 202–215 (1948).

The method of parts I and II [see the preceding review] is extended to open-wire structures. The first prototype has two equal coplanar strips and leads directly to $z=d$ sn (u, k) with $k=d/(a+d)$, where a is the strip width and $a+2d$ is the distance of strip centers. The case of strips of different widths involves tandem mapping: $z'(z(u))$, with $z'(z)$ linear. The other prototype of two equal strips in opposite position is reduced to the first one via tandem mapping, $z'(z)$ involving a Jacobian integral of the second kind and thus, for $z'(u)$, the Jacobian ζ-function. Similarly for unequal widths. The quantities Z and β are computed for the various finite strip configurations. For β this involves the assumption of small, but finite, radius of curvature, lest the integral diverge; the latter becomes tractable on application of the logarithmic approximations of elliptic integrals for small complementary moduli. [Reviewer's note. The validity of this procedure remains in question without a limit consideration involving current density distribution (penetration) at small radii of curvature.] The values of Z thus computed are plotted versus the width-to-distance ratio together with the elementary homogeneous field approximation.

H. G. Baerwald (Cleveland, Ohio).

King, Ronold. The theory of N coupled parallel antennas. J. Appl. Phys. 21, 94–103 (1950).

The integral-equation theory of coupled antennas developed mainly by the author and his collaborators for two- and three-coupled antennas is generalized to any number of units symmetrically arranged around a circle. All units are parallel, nonstaggered, and fed or loaded at the centre. Complications arising from transmission-line connections are eliminated by the introduction of slice generators. Various practical antenna systems are briefly considered.

C. J. Bouwkamp (Eindhoven).

Graffi, Dario. Sulla propagazione delle onde elettromagnetiche in un tubo curvo. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 5 (1947/48), 23–26 (1949).

Coleman, B. L. Propagation of electromagnetic disturbances along a thin wire in a horizontally stratified medium. Philos. Mag. (7) 41, 276–288 (1950).

Let two semi-infinite conducting media such as air and earth have a plane interface. A wire runs parallel to that interface. At one point a sinusoidal current of given frequency ω is sent into the wire and this current leaks off into the medium. The attenuation of the current along the wire in the steady alternating state is found here when the wire is a slender one, in the two cases in which the wire does or does not lie at the interface. The result is useful because operational processes such as the method of using Fourier integrals can be employed to compute the current along the wire corresponding to surges of the applied current such as those produced by lightning. Using Maxwell's theory, Fourier transforms and residues the author concludes that for slender wires the attenuation factor for the current is of the type e^{-cd} where d denotes the distance along the wire from the point of application of the current. When the wire lies several diameters away from the interface c depends in a simple way upon the electromagnetic coefficients of the medium in which the wire is buried and upon the frequency ω. If the wire lies at the interface c is given in terms of ω and the coefficients of the two media. Cases of more than two layers are considered.

R. V. Churchill.

Westfold, K. C. The wave equations for electromagnetic radiation in an ionized medium in a magnetic field. Australian J. Sci. Research. Ser. A. 2, 169–183 (1949).

This is an investigation of the characteristics of plane waves and plasma oscillations which can exist in a magneto-ionic medium. The wave equation for the field is solved for the case where D and E are in the same direction. It seems that three types of plasma oscillation are possible in the electron plasma of given density, collision frequency and magnetic field. The effect of the collision seems to be to damp the oscillations and to reduce the natural frequencies.

R. Truell (Providence, R. I.).

Bullard, E. C. Electromagnetic induction in a rotating sphere. Proc. Roy. Soc. London. Ser. A. 199, 413–443 (1949).

The boundary value problem of a rotating conducting sphere surrounded by a stationary conducting shell is discussed and applications to terrestrial magnetism are considered.

C. Kikuchi (East Lansing, Mich.).

Toraldo di Francia, G. A variational principle for the computation of reflection coefficients. Physical Rev. (2) 78, 298 (1950).

The author remarks that variational principles may be used for calculating the reflection coefficient of a potential barrier. An example is discussed.

A. E. Heins.

Schaefer, Clemens, und v. Fragstein, Conrad. Zur Theorie der Reflexion und Brechung. Ann. Physik (6) 6, 39–43 (1949).

Miles, J. W. Errata: On the diffraction of an electromagnetic wave through a plane screen. J. Appl. Phys. 21, 468 (1950).

Cf. the same J. 20, 760–771 (1949); these Rev. 11, 141.

Bianchi, Vittorio. Sulla teoria matematica dei fenomeni elettrodinamici dedotta unicamente dall'esperienza. Elettrotecnica 36, 220–246 (1949).

In its original form Ampère's law of force between two current elements was based on experimental results, together with the assumption that the force between any two elements acted along the line joining their midpoints. It has long been realized that the assumption is false, but in the present paper the author discusses the whole question at great length. Then he derives a formula for the law of force based entirely on two fundamental laws which may be experimentally verified. The first law states that the force on a rectilinear current-carrying conductor in a magnetic

field is always perpendicular to the conductors; while the second states that there is no force between collinear current elements. The resulting formula is identical with that usually given in modern text books, except in form. The Neumann integral for the mutual inductance between current-carrying conductors is criticised in somewhat similar fashion, and a formula for the mutual inductance between arbitrarily oriented current elements is derived. This is used to evaluate the self and mutual inductances of circular coils in various alignments. *M. C. Gray.*

Havas, Peter. Bemerkungen zum Zweikörperproblem der Elektrodynamik. Acta Physica Austriaca 3, 342–351 (1950).

Two problems in electrodynamics are discussed. One concerns the motion of two particles with charges of opposite sign in an external field. It is shown that there exist orbits for the two particles such that every point on one lies outside the light-cone of every point on the other. Hence the two particles move completely independently. A simple example is the case of a positive and a negative particle moving along the same direction (x) in a uniform electric field. The path of each particle is a hyperbola in the (x, t)-plane; if the centers of the two hyperbolae are spacelike to one another, the entire orbits are.

Classical one-particle models of pair creation and annihilation are discussed in a second section. They are based on the fact that two particles of opposite charge that annihilate can be replaced by one particle, provided the proper time of part of the orbit decreases instead of increasing [R. P. Feynman, Physical Rev. (2) **74**, 939–946 (1948); M. Schönberg, ibid. **69**, 211–224 (1946); E. C. G. Stückelberg, Helvetica Phys. Acta **14**, 588–594 (1941); these Rev. **10**, 222; **8**, 428; **4**, 56]. This formal reduction to a one-particle problem does not result in a real solution of the difficulties of interacting particles, because different parts of such an orbit will be inside one another's light-cones; hence retarded and/or advanced interactions, with their usual difficulties, must be taken into account.

R. Karplus (Cambridge, Mass.).

Schlomka, Teodor. Die elektrischen und magnetischen Flächenwirbel bei bewegten Körpern. Ann. Physik (6) **5**, 190–196 (1949).

Let $x', \cdots, t'$ refer to coordinates fixed relative to contiguous rigid bodies K_i moving with common linear velocity $\mathbf{v}$, and having a common boundary S. Let $\mathbf{F}=\mathbf{F}(x', \cdots)$, independent of t', be a vector field continuous within the K_i but discontinuous across S. Then, if $x, \cdots, t$ refer to a resting observer, $\mathbf{F}(x, \cdots, t)$ will be discontinuous across the (moving) surface S. If $\Delta\mathbf{F}=\mathbf{F}_1-\mathbf{F}_2$, it is found that $\Delta\mathbf{F}/\partial t=(\mathbf{F}_1-\mathbf{F}_2)v_n$. This result is variously applied to the case where $\mathbf{F}$ is electromagnetic, yielding formulas like

$$\operatorname{Rot}\mathbf{H}=\mathbf{j}_{fl}+(\mathbf{D}_1-\mathbf{D}_2)v_n,$$
$$\operatorname{Rot}\mathbf{E}=-(\mathbf{B}_1-\mathbf{B}_2)v_n,$$

in which $\mathbf{j}_{fl}$ is the surface current on S, and similarly Rot refers to S. *A. L. Foster* (Berkeley, Calif.).

Flamm, Ludwig. Elektrische Feldmechanik. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **156**, 175–202 (1948).

This paper is identical with one already reviewed [Acta Physica Austriaca 1, 259–284 (1948); these Rev. **9**, 551].

M. C. Gray (Murray Hill, N. J.).

Ivanović, D. Über die Bewegungsgleichungen geladener Teilchen im elektromagnetischem Felde. Bull. Soc. Math. Phys. Serbie 1, no. 3-4, 59–72 (1949). (Serbian. German summary)

Der Verf. gibt eine Übersicht der Grundgleichungen der Bewegung geladener Teilchen im elektromagnetischen Felde.

From the author's summary.

Sengupta, N. D. On the scattering of electromagnetic waves by free electron. I. Classical theory. Bull. Calcutta Math. Soc. **41**, 187–198 (1949).

The scattering of electromagnetic waves by a free electron is dealt with classically using the Lorentz equation. The relativistic equation of motion is solved and the scattered radiation is determined from the Liénard-Wiechert potential corresponding to the velocity of the electron obtained from the solution of the equation of motion. The scattered radiation is composed of a spectrum of frequencies which are all integer multiples of the basic frequency. The usual Thompson scattering is the first order scattering. The author makes no claim to anything new but has derived these results to compare with corresponding formulas from quantum mechanics to be offered later as the second part of this paper. *R. Truell* (Providence, R. I.).

Schubert, Gerhard U. Der Energie-Impulstensor in der von Laue-Londonschen Elektrodynamik des Supraleiters. Ann. Physik (6) **6**, 163–168 (1949).

The conservation laws for momentum and energy of the electrodynamics of the superconductor [F. London and H. London, Proc. Roy. Soc. London. Ser. A. **149**, 71–88 (1935); v. Laue, same Ann. (5) **42**, 65–83 (1942); **43**, 223–224 (1943); these Rev. **5**, 162, 279] can be written as a relation between quantities which are Lorentz invariant. The force density in a superconductor is derived from the generalized momentum-energy tensor. *F. W. London* (Durham, N. C.).

Quantum Mechanics

Vrkljan, Vladimir S. On the transition of the Dirac's equations in the Maxwell's equations. Hrvatske Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **4**, 104–112 (1949). (Croatian. English summary)

Brittin, Wesley E. A note on the quantization of dissipative systems. Physical Rev. (2) **77**, 396–397 (1950).

E. Kanai [Progress Theoret. Physics 3, 440–442 (1948)] has shown that it may be possible to put the equations of motion of a dissipative system into the Hamiltonian form and then quantize them. However, in the examples which he treated the Hamiltonian functions depended explicitly on the time and the commutation relations led to a violation of the usual uncertainty principle as applied to a coordinate and a velocity. In this paper the author investigates the possibility of quantizing a dissipative system without introducing an explicit dependence on the time. By starting with the classical equations of motion and carrying out a transition to the quantum theory, he finds that in the Heisenberg representation the only dissipative forces consistent with the formalism are those which are functions of the coordinates alone. On the other hand, in the Schrödinger representation he finds that no dissipative forces are possible.

[There is an error in the discussion of the latter case, but it is not serious since it does not invalidate the conclusion.]
N. Rosen (Chapel Hill, N. C.).

Szamosi, G. Über die Verallgemeinerung der kanonischen Feldgleichungen. Hungarica Acta Physica 1, no. 6, 27–31 (1949).

A prescription is given for obtaining the Hamiltonian field equations and the canonically conjugate variables from a Lagrangian which depends on high deviates of the field variables. The method is a generalization of Ostrogradski's treatment of the same problem in classical particle mechanics [Mém. Acad. Imp. Sci. Saint-Pétersbourg (6) 6, Première Partie. Sci. Math. Phys. 4, 385–517 (1850)]. It is based on treating all time derivatives of the field variables, except the highest, as independent canonical variables. Their conjugate momenta are derived from the Lagrangian in the usual way. Bose-Einstein or Fermi-Dirac quantization then leads to a quantum field formalism in agreement with several others that have recently been described [T. S. Chang, Proc. Cambridge Philos. Soc. 42, 132–138 (1946); 43, 196–204 (1947); J. S. de Wet, ibid. 44, 546–559 (1948); Proc. Roy. Soc. London. Ser. A. 195, 365–376 (1948); these Rev. 7, 404; 8, 426; 10, 91, 418]. The proof of relativistic invariance given by these two authors therefore applies.
R. Karplus (Cambridge, Mass.).

Dirac, P. A. M. La seconde quantification. Ann. Inst. H. Poincaré 11, 15–47 (1949).

In this article the fundamental work of Fock [Z. Physik 49, 339–357 (1928); Phys. Z. Sowjetunion 6, 425–469 (1934)] is reviewed and the mathematical identity between the theory of a system of like particles satisfying the Einstein-Bose statistics, a system of bosons, and a system of harmonic oscillators is shown. The theory of a system of particles obeying the Fermi-Dirac statistics is also given. The relativistic form of these theories is discussed and the second quantization of the Maxwell equations is formulated.
A. H. Taub (Urbana, Ill.).

Valatin, Jean-G. L'algèbre extérieure et la seconde quantification. C. R. Acad. Sci. Paris 230, 722–724 (1950).

The author observes that the creation and absorption operators of the second quantization can be simply expressed in terms of Grassmann multiplication. *I. E. Segal.*

Nikol'skiĭ, K. V. On infinite matrices used in the theory of second quantization. Doklady Akad. Nauk SSSR (N.S.) 72, 39–40 (1950). (Russian)

Let $\Omega_n=a_1e_{12}+a_2e_{23}+\cdots+a_{n-1}e_{n-1,n}+e_{n1}$ be an $n\times n$ matrix, where e_{ij} is the matrix unit with 1 in the i, j position and 0 elsewhere. If the infinite product $\prod a_i$ converges, then $\lim_{n\to\infty}(\Omega_n)^n$ is the infinite scalar matrix $\prod a_i$. This gives a matric-theoretic interpretation of infinite convergent products. *J. L. Brenner* (Pullman, Wash.).

Coester, F., and Jauch, J. M. On the role of the subsidiary condition in quantum electrodynamics. Physical Rev. (2) 78, 149–157 (1950).

This paper gives a relativistically invariant definition of the vacuum state in the Tomonaga-Schwinger theory of the photon field. To this end, operators $\mathfrak{A}_\mu(x)$ are defined in terms of a time-like unit vector n_μ, and the usual field operators $A_\mu(x)$. The operator $\mathfrak{A}_\mu^{(+)}(x)$ is the positive frequency part of $\mathfrak{A}_\mu$. The authors take (1) $\mathfrak{A}_\mu^{(+)}(x)\Phi_0=0$ and (2) $\partial^\mu A_\mu(x)\Phi_0=0$ as the conditions on a solution Φ_0 of Schrödinger's equation defining the vacuum. It is shown that these conditions are independent of n_μ by an argument similar to that of Belinfante [Physical Rev. (2) 76, 226–233 (1949); these Rev. 11, 146]. Condition (1) implies the absence of transverse photons; (2) is the "subsidiary condition" and along with (1) rules out longitudinal photons. Incidental to the main argument, the authors show that the appearance of a Coulomb interaction term in Schrödinger's equation has nothing to do with the subsidiary condition, a fact which they assert had not been made clear by previous treatments.

Footnote (11) is of considerable mathematical interest since it points out that a complete discussion of the problem would require a proof that a solution of Schrödinger's equation satisfying (1) and (2) exists and is unique. Certainly Φ_0 is not a vector in Hilbert space. Since the authors remark that "the definition of the vacuum presents a difficulty which contains the essential features of the difficulties common to all problems in quantum electrodynamics," it would seem that the central mathematical problem of current field theory is how to enlarge the concept of a state vector so that the above problem has a solution and how to restrict the concept so that the above problem has only one solution. By employing two different script A's the authors invited a series of typographical errors which make § III of their paper almost unintelligible. Definitions (32) and (46) should be of different symbols [cf. the authors' erratum note in the same vol., 827 (1950)].
A. J. Coleman (Toronto, Ont.).

Umezawa, Hiroomi, and Kawabe, Rokuo. An improvement on the integrations appearing in perturbation theory. Progress Theoret. Physics 4, 420–422 (1949).

The authors propose to modify the use of perturbation theory in quantum mechanics, in such a way as to make the calculations formally covariant; their method applies to radiation theory in its standard non-covariant form and avoids the introduction of the Tomonaga-Schwinger formalisms. Let the integrals over intermediate states appearing in perturbation theory be written formally as integrals over all possible intermediate states without regard to momentum conservation. Then one or more scalars W are constructed which are functions (e.g., scalar products) of the momenta and energies of the particles in the intermediate states. Integration is carried out first over all other variables, and last over the variables W. In this way divergent self-energies and vacuum polarizations appear in correctly covariant form. The method will be applied to various specific problems in subsequent papers. *F. J. Dyson.*

Umezawa, Hiroomi, and Kawabe, Rokuo. Some general formulae relating to vacuum polarization. Progress Theoret. Physics 4, 423–442 (1949).

Formulae for the second-order effects of the polarization of the vacuum are written down in the notations of perturbation theory. The formulae apply to the electrodynamical properties of charged particles of spin 0, $\frac{1}{2}$, or 1. The effects are considered in detail for two particular cases, the elastic scattering of two charged particles, and the scattering of a photon by a particle. The formulae are long and unwieldy; their reduction to manageable form is left for the following paper. *F. J. Dyson* (Princeton, N. J.).

Umezawa, Hiroomi, and Kawabe, Rokuo. Vacuum polarization due to various charged particles. Progress Theoret. Physics **4**, 443–460 (1949).

Using the formulae of the preceding paper, combined with the method of evaluation described in the second preceding paper [see the two preceding reviews], the second-order polarizations of the vacuum due to charged scalar, spinor and vector fields are calculated. All these quantities have the correct covariance properties, not only in the leading divergent terms but in the finite terms as well. The results for the spinor case are in agreement with those of Schwinger [Physical Rev. (2) **75**, 651–679 (1949); these Rev. **10**, 662]. In the vector case, a logarithmically divergent term appears which cannot be removed by mass and charge renormalization. By an extension of the work here described, it is clear that an unambiguous separation of finite from infinite effects could be made by the use of the authors' methods of calculation with standard non-covariant perturbation theory, in every situation where such a separation could be made using the covariant formalisms of Tomonaga and Schwinger.

F. J. Dyson (Princeton, N. J.).

Kawabe, Rokuo, and Umezawa, Hiroomi. The self-energy of a Dirac particle, and its relativistic covariance. Progress Theoret. Physics **4**, 461–467 (1949).

This paper applies an idea proposed by the authors elsewhere [see the three preceding reviews] to calculate the self-energy of a Dirac particle. The infinite self-energy due to vacuum polarization is compensated by interaction energy with a C-meson. The essential step in the present method of calculating these energies is to effect the integration in momentum space by means of an invariant scalar quantity and so ensure relativistic invariance. For example, the electromagnetic self-energy is evaluated by means of a quantity proportional to the scalar product of the energy momentum four-vectors of two particles appearing in intermediate states.

The formula obtained for the electromagnetic self-energy coincides with that of Schwinger [Physical Rev. (2) **75**, 651–679 (1949); these Rev. **10**, 662]. In calculating the interaction energy with the C-meson, three cases are distinguished according as δ, the ratio of the masses of the C-meson and the Dirac particle, is positive, zero or negative. For $\delta<0$, the result obtained agrees with that of Pais [Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. 1. **19**, no. 1 (1947); these Rev. **8**, 554]. When the Dirac particle is ascribed the mass of a proton, the resulting energy is interpreted as the mass difference between the proton and the neutron regarded as two states of a single nucleon differing only in e and f charges. Only 65% of the observed mass difference is accounted for in this way. *A. J. Coleman.*

de Broglie, Louis. Une conception nouvelle de l'interaction entre les particules chargées et le champ électromagnétique. Portugaliae Math. **8**, 37–58 (1949).

The self-energy of a point electron interacting with the electromagnetic field can, as is well known, be made finite by subtracting from it the self-energy arising from the interaction of the electron with a hypothetical neutral vector meson field of short range. [F. Bopp, Ann. Physik (5) **38**, 345–384 (1940); R. P. Feynman, Physical Rev. (2) **74**, 939–946 (1948); these Rev. **2**, 336; **10**, 222.] Previously no consistent theory of the subtractive vector field had been discovered; the subtraction procedure remained a mathematical device without physical justification. The author claims to have developed a formalism including in a natural way vector fields with the subtractive property. The present article is an elementary survey of the author's method, referring back to earlier papers for technical details. The reviewer was unable to judge whether the author's claim is justified. *F. J. Dyson* (Princeton, N. J.).

Stueckelberg, E. C. G., et Rivier, D. A propos des divergences en théorie des champs quantifiés. Helvetica Phys. Acta **23**, 236–239 (1950).

Earlier papers [Rivier, Helvetica Phys. Acta **22**, 265–318 (1949); these Rev. **11**, 301; Stueckelberg and Rivier, ibid. **23**, 215–222 (1950); these Rev. **11**, 569] are supplemented by a discussion of the divergencies and the ambiguities encountered there. It is proposed to define "true" nuclei of integration as those obtained by removing the nonintegrable singularities from the nuclei of integration which appear in solutions of the Hamiltonian differential equations. This removal is achieved by the use of a suitable operator. It leads to the introduction of arbitrary constants whose number depends upon the character of the singularity. The second order self-energy of the photon and of the meson can be made zero by a suitable choice of these arbitrary constants. Similarly, it is also possible to avoid a renormalization of the electronic charge in the second order approximation.

E. Gora (Providence, R. I.).

Sawada, K. A divergence-free field theory. Progress Theoret. Physics **4**, 374–375 (1949).

The author proposes to eliminate divergences in quantum electrodynamics by the following formal device. Consider for simplicity an electron interacting with a scalar field ϕ representing mesons of mass μ. The usual commutation relation for ϕ is

$$(1)\qquad [\phi(x), \phi(x')]=i\Delta(x-x', \mu^2).$$

Suppose that we introduce for every value of λ a field $\phi(\lambda^2, x)$ subject to the conditions

$$(2)\qquad \phi(\mu^2, x)=\phi(x),$$

$$(3)\qquad [\phi(\lambda^2, x), \phi(\lambda'^2, x')]=i\Delta(x-x', \lambda\lambda').$$

Then we may write

$$(4)\qquad \phi(x)=-\int_\mu^\infty (\partial\phi(\lambda^2, x)/\partial\lambda)d\lambda.$$

From (3) and (4) the second-order self-energy of an electron interacting with the ϕ-field may be calculated, and its value is finite.

This method of removing divergences is closely related to the regulation method of Feynman [Physical Rev. (2) **74**, 939–946, 1430–1438 (1948); these Rev. **10**, 222, 345] and Pauli and Villars [Rev. Modern Physics **21**, 434–444 (1949); these Rev. **11**, 301]. It would have some advantage over the latter methods, if a consistent interpretation of the $\phi(\lambda^2, x)$ fields and their commutation-relations (3) could be given. The question of consistency is not discussed by the author. He reserves for later publication the discussion of other divergences, photon self-energy and vacuum polarization. *F. J. Dyson* (Princeton, N. J.).

Sawada, Katuro. A divergence free field theory. Progress Theoret. Physics **4**, 412–419 (1949).

This paper sets out more fully the work described in the preceding review. *F. J. Dyson* (Princeton, N. J.).

Sawada, Katuro. Note on the finite extension of electron. Progress Theoret. Physics 4, 275–286 (1949).

Using Dirac's positron theory, Weisskopf [Physical Rev. (2) 56, 72–85 (1939)] has calculated the charge distribution of the electron in connection with the self-energy problem. The author completes this study by considering also the extension of the electron due to forced oscillations under the influence of the zero-point fluctuations of the radiation field. The average "radius" of the electron turns out to be finite and equal to its Compton wave-length for interaction with the electromagnetic field, while Weisskopf's "radius" tends to zero logarithmically with the cut-off frequency at high energies. It is shown (independently of Welton) that this fluctuation effect gives rise to finite corrections for reaction problems, and a non-relativistic formula for the electromagnetic shift of energy levels is derived.

E. Gora (Providence, R. I.).

Hu, Ning. On the treatment of quantum electrodynamics without eliminating the longitudinal field. Physical Rev. (2) 77, 150 (1950).

This note extends and corrects a few remarks appearing in an earlier paper by the same author [same Rev. (2) 76, 391–396 (1949); these Rev. 11, 146]. *C. Kikuchi.*

Sasaki, Muneo, and Suzuki, Ryoji. On the reaction of radiation field. Progress Theoret. Physics 4, 485–491 (1949).

The authors calculate the shift in the energy levels caused by second-order radiative corrections, for a hypothetical hydrogen atom consisting of a charged spin-zero particle in a Coulomb field. The calculation is covariant, based on the covariant formulation of spin-zero electrodynamics of Kanesawa and Tomonaga [same journal 3, 1–13 (1948); these Rev. 10, 227]. The shift obtained is finite, and differs slightly from the value obtained by the reviewer in a non-covariant calculation [Physical Rev. (2) 73, 617–626 (1948)]. The difference is due to inadequacies in the reviewer's method. *F. J. Dyson* (Princeton, N. J.).

Jost, Res, und Luttinger, J. M. Vacuumpolarisation und e^4-Ladungsrenormalisation für Elektronen. Helvetica Phys. Acta 23, 201–214 (1950).

The authors investigate the fourth-order radiative corrections to the interaction of two charges in quantum electrodynamics. The charges are taken to be external classical charges, not electrons or positrons. Thus the radiative corrections are due simply to the polarization of the electron-positron vacuum by the external charges. There are two types of divergence in quantum electrodynamics which can be removed by compensation, by introducing into the theory additional hypothetical short-range fields which produce effects of the opposite sign. These removable divergences are the electron and the photon self-energies. On the other hand, the divergent charge-renormalization produced by vacuum polarization gives always an induced charge which decreases the effect of the external charge, and so is of the same sign for all types of field and cannot be removed by compensation. In order to see whether there was any possibility of saving the idea of compensation as a method of removing all divergences, the authors undertook the calculation of the fourth order contribution to charge-renormalization, in the hope that it might be of the opposite sign to the second-order contribution. If the two contributions had opposite sign, then they could perhaps compensate each other for a certain special value of the fine-structure constant $\alpha=(e^2/\hbar c)$. The values which are obtained for the charge-renormalizations of second and fourth order are respectively

$$-(\alpha/3\pi)\log,\quad -(\alpha^2/4\pi^2)\log,$$

where log is a certain logarithmically divergent integral. Thus no compensation is possible.

In the course of the rather difficult calculations the authors have shown how fourth-order radiative corrections in electrodynamics are to be consistently handled using the ideas of mass and charge renormalization. They have cleared up several delicate points of technique which arise in considering such high-order processes. *F. J. Dyson.*

Bhabha, H. J. On the postulational basis of the theory of elementary particles. Rev. Modern Physics 21, 451–462 (1949).

Precise forms of postulates are given as the basis of present relativistic quantum theory concerning relativistic invariance, the existence of a deterministically propagated wave-function ψ, homogeneous isotropic space-time, a Lagrangian L rational and integral in ψ and its derivatives. The most general quadratic terms in ψ occurring in L give a wave equation $(\alpha^k p_k+\beta\chi)\psi=0$: charge-current vector and energy-momentum tensor are constructed. From interaction terms and essential non-linearities arises a criterion for an elementary physical entity according to which an electron and positron are different states of the same entity while states of different mass of a physical entity are different elementary physical entities. Every particle must have an antiparticle. For particles of half-odd-integral spin neither energy density nor total energy can be positive definite and neither charge density nor total charge for particles of integral spin. The minimal equation for the α gives rise to commutation relations for them and to statements about the rest mass. For a minimal equation of degree two the Dirac equation is unique. Equations for particles of spins 0 and 1 are discussed. *C. Strachan* (Aberdeen).

Bhabha, H. J. On a new theory of nuclear forces. Physical Rev. (2) 77, 665–668 (1950).

This paper applies the author's general theory of elementary particles [see the preceding review] to fields corresponding to particles with only one possible mass value. For a slowly moving point charge the theory then admits a potential of the form $1/(2\chi)e^{-\chi r}$, where χ is the rest-mass of the particle. The dipole-dipole potential follows from this in the usual fashion and has the advantage that the singular r^{-3} term, which occurs in the Yukawa theory, does not appear. *A. J. Coleman* (Toronto, Ont.).

Matthews, P. T. The S-matrix for meson-nucleon interactions. Philos. Mag. (7) 41, 185–195 (1950).

The author claimed to prove that a divergence-free S-matrix could be obtained, giving finite matrix elements for all scattering processes between free particles, for the quantized theory of a nucleon field interacting either with a scalar meson field with scalar interaction or with a pseudoscalar meson field with pseudoscalar interaction. The proof was on the lines of the reviewer's discussion of electrodynamics [Physical Rev. (2) 75, 1736–1755 (1949); these Rev. 11, 145]. The author has pointed out [private communication to the reviewer] that the proof is erroneous. It is an open question whether the result stated is true.

F. J. Dyson (Birmingham).

Hjalmars, Stig. On the general formulation of meson pair theory. Ark. Fys. 1, 41–116 (1949).

The author formulates the most general pair interaction between nucleons and charged scalar and vector mesons. The paper contains a review of Dirac equations for particles of arbitrary spin which includes a very convenient representation of the Kemmer-Duffin β matrices.

A. H. Taub (Urbana, Ill.).

Kanesawa, Suteo, and Koba, Zirô. A remark on relativistically invariant formulation of the quantum field theory. Progress Theoret. Physics 4, 297–311 (1949).

Covariant quantized field theories can be constructed in two ways. Schwinger [Physical Rev. (2) 74, 1439–1461 (1948); these Rev. 10, 345] starts from a Hamiltonian formalism with field variables and equations of motion first defined in a Heisenberg representation. Tomonaga [same journal 1, 27–42 (1946); these Rev. 10, 226] defines field variables and postulates his equation of motion

$$\{L_P(C)+i\hbar(\delta/\delta C_P)\}\Psi(C)=0 \tag{1}$$

directly in the interaction representation. The extension of the Schwinger method to include all field-theory formalisms of the canonical Hamiltonian type has been worked out by Kroll [Physical Rev. (2) 75, 1321 (1949)], Roberts [ibid. (2) 77, 146–147 (1950)], and others. By general arguments these authors show that, given any classical field-theory in canonical form, a corresponding covariant quantized theory can be set up; the consistency of the quantized theory is automatic. The Tomonaga method, on the other hand, has been extended to meson theories by Kanesawa and others [Kanesawa and Tomonaga, same journal 3, 1–13, 101–113 (1948); these Rev. 10, 227] only by considering each theory separately; in each case the theory was proved consistent by verifying the integrability of (1), the $L_P(C)$ being chosen so as to satisfy the integrability condition

$$\frac{\delta L_X(C)}{\delta C_Y}-\frac{\delta L_Y(C)}{\delta C_X}=-\frac{i}{\hbar}[L_X(C),L_Y(C)]. \tag{2}$$

The present paper gives a general prescription for determining $L_P(C)$ for any field theory, given the interaction Lagrangian density $L(P)$, whether a canonical formalism exists or not. It is proved that the $L_P(C)$ so found will coincide with the interaction Hamiltonian derived from the canonical formalism whenever the latter exists. The method of determining $L_P(C)$ is to find an explicit solution of (2); the method is complicated, and the Hamiltonian method, when it is applicable, is much more convenient. But it is of importance that a method now exists for constructing non-canonical covariant field theories.

In the last section the method is applied to construct a covariant formalism for two spinor fields ϕ and ψ with an interaction density

$$L(P)=g(\bar{\phi}\gamma_\mu\phi)((\partial/\partial x_\mu)(\bar{\psi}\psi)).$$

This is the interaction in the theory of β-decay of Konopinski and Uhlenbeck [Physical Rev. (2) 48, 7–12 (1935)], a theory for which no canonical formalism exists. The Tomonaga formalism can be set up very easily, the correct $L_P(C)$ being

$$g(\bar{\phi}\gamma_\mu\phi)\frac{\partial}{\partial x_\mu}(\bar{\psi}\psi)-\frac{2i}{\hbar c}g^2(\bar{\phi}N_\mu\gamma_\mu\phi)^2\bar{\psi}\psi,$$

where N_μ is the unit vector normal to C. *F. J. Dyson.*

Kanesawa, S. Quantum theory of generalized local fields. Progress Theoret. Physics 5, 157–158 (1950).

The author [same journal 4, 238–239 (1949)] and the author and Z. Koba [ibid. 4, 297–311 (1949); see the preceding review] proposed a method of setting up an interaction representation for field theories with more general types of interaction than can be handled by current methods. They now show that the condition of integrability of the Schrödinger equation in their formalism is identically the same as the condition for conservation of a suitably defined energy-momentum tensor. This observation does not eliminate from their method the practical difficulty that there is no simple rule for finding the correct form of the interaction Hamiltonian. *F. J. Dyson* (Princeton, N. J.).

Feynman, R. P. Space-time approach to quantum electrodynamics. Physical Rev. (2) 76, 769–789 (1949).

"In this paper two things are done. (1) It is shown that a considerable simplification can be attained in writing down matrix elements for complex processes in electrodynamics. Further, a physical point of view is available which permits them to be written down directly for any specific problem. Being simply a restatement of conventional electrodynamics, however, the matrix elements diverge for complex processes. (2) Electrodynamics is modified by altering the interaction of electrons at short distances. All matrix elements are now finite, with the exception of those relating to problems of vacuum polarization. The latter are evaluated in a manner suggested by Pauli and Bethe, which gives finite results for these matrices also. The only effects sensitive to the modification are changes in mass and charge of the electrons. Such changes could not be directly observed. Phenomena directly observable are insensitive to the details of the modification. For such phenomena, a limit can be taken as the range of the modification goes to zero. The results then agree with those of Schwinger. The methods apply as well to charges obeying the Klein-Gordon equation, and to the various meson theories of nuclear forces. Illustrative examples are given. Although a modification like that used in electrodynamics can make all matrices finite for all meson theories, for some theories it is no longer true that directly observable phenomena are insensitive to the details of the modification.

"The paper is a continuation of a preceding one [same vol., 749–759 (1949)] in which the motion of electrons neglecting interaction was analyzed. Here the same technique is applied to include interaction. The genesis of the theory was this. The conventional electrodynamics was expressed in the Lagrangian form of quantum mechanics [Rev. Modern Physics 20, 367–387 (1948); these Rev. 10, 224]. This was not complete because the Lagrangian method had been worked out only for particles obeying the non-relativistic Schrödinger equation. It was then modified in accordance with the requirements of the Dirac equation and the phenomenon of pair creation. Finally for practical calculations the expressions were developed in a power series in $(e^2/\hbar c)$. Each term in the series had a simple physical interpretation. Since the result was easier to understand than the derivation, it was thought best to publish the results first. The derivation will appear in a separate paper." From the author's summary.

This paper provides the necessary information for the application of the author's radiation theory to practical problems. For many problems this theory gives the simplest and most convenient solution at present available. The

derivation of the theory from the Lagrangian form of quantum mechanics being still unpublished, the arguments justifying the author's methods are intuitive and not always convincing. Especially this is true of the arguments leading to the introduction of the functions K_+ and δ_+ which play a decisive role in the formalism. The functions have been independently introduced by E. C. G. Stueckelberg [Rivier and Stueckelberg, same Rev. (2) **74**, 218 (1948) and references there quoted] whose use of them is also somewhat intuitive. Basically, the role of the functions is to preserve the principle of causality which is in danger of being lost when the differential Hamiltonian form of quantum mechanics is abandoned. But a precise statement of this idea has not yet been formulated. *F. J. Dyson.*

Nambu, Yôichirô. The use of the proper time in quantum electrodynamics. I. Progress Theoret. Physics **5**, 82–94 (1950).

A theory has been proposed by R. P. Feynman [Physical Rev. (2) **76**, 749–759 (1949); cf. the paper reviewed above] in which electrons and positrons are described by world-lines of a single type of particle, the electrons corresponding to sections of the world-line moving forward in time, the positrons to sections of the world-line moving backward in time. In this theory the equation of motion of a particle is expressed in terms of a parameter, the proper time, which increases monotonically along the world-line. The "real" time is thus not a monotonic function of the proper time, and the proper time is not a one-valued function of the real time.

The author here proposes to introduce a new parameter into the formulation of quantum mechanics, to which he gives the name of proper time. It is to refer not to an individual particle but to the whole system. The fundamental equation of motion is taken to be $i\partial\Psi/\partial\tau=(i\partial/\partial t-H)\Psi$, where τ is the proper time and t the ordinary time. He develops the new formalism far enough to show that in some circumstances it gives results in agreement with the usual theory. It is not clear to the reviewer whether the author intends merely a reformulation of existing quantum theory, or a change in its physical content. *F. J. Dyson.*

Nambu, Yôichirô. On the method of the third quantization. Progress Theoret. Physics **4**, 331–346 (1949).

The developments of quantum electrodynamics due to Tomonaga [same journal **1**, 27–42 (1946); these Rev. **10**, 226] and Schwinger [Physical Rev. (2) **75**, 651–679 (1949); these Rev. **10**, 663], require the separation of the one, two, three, etc. particle portions of multilinear expressions of quantized field operators. This paper is an attempt to formalize the procedures used. Using the known commutation relations, any product of field operators may be arranged with all emission operators to the left of those for absorption. Such "well ordered" expressions are considered as wave functions. Multiplication with another field operator yields a new well ordered "wave function." This may be expressed in terms of an operator of a different kind ($q-3$ numbers) acting on the "wave function." Such operators are constructed and an explicit matrix representation is given. To construct a basis for the theory of such operators two different possible definitions of the scalar product of wave functions in the Hilbert space of these operators are given. As examples, the transformations between the different possible representations for the equations of quantum electrodynamics are found. In particular, the transformation from the Heisenberg representation to an "interaction" representation in which the Schrödinger functional varies as for no interaction is obtained. *K. M. Case.*

Nambu, Yôichirô. On the method of the third quantization. II. Progress Theoret. Physics **4**, 399–411 (1949).

The formal methods of the paper reviewed above are applied to the specific problem of radiative corrections in electrodynamics. Using a new interaction representation the second order corrections to the electron and electromagnetic field operators are obtained. Similarly, the corrected energy and current operators are found and the various physical effects separated. The results of Schwinger [Physical Rev. (2) **75**, 651–679 (1949); these Rev. **10**, 663] are found with slightly less calculation. Other applications to non-localized field theories and the obtaining of the generating functions for conventional eigenvalue problems are suggested. *K. M. Case* (Princeton, N. J.).

Balseiro, José A. Transformation of configurations of a radiation field. Application to the radiation of multipoles. Revista Unión Mat. Argentina **14**, 64–78 (1949). (Spanish. English summary)

The problem of determining the photon distribution over the states of a quantized radiation field described by means of two systems of orthogonal vibrations both referred to the same radiation field is solved. The formalism is applied in order to find the probabilities of a given configuration of the radiation emitted by electric and magnetic multipoles. Known expressions for angular intensity distribution are obtained. *Author's summary.*

Le Couteur, K. J. The interaction of point particles with charged fields. Proc. Cambridge Philos. Soc. **45**, 429–440 (1949).

This is a classical theory of the interaction of a nucleon with a charge bearing field. For the latter the symmetrical vector meson field is chosen and both vector and tensor coupling are assumed. Treating the spin and isotopic spin of a nucleon as classical angular-momentum-like unit vectors, the equations of motion for the problem of one nucleon in an external classical mesonic radiation field are integrated, following closely Harish-Chandra's treatment for a neutral meson coupling [Proc. Roy. Soc. London. Ser. A. **185**, 269–287 (1946); these Rev. **7**, 538]. The possibility of self-oscillations of the isotopic spin (existence of charge isobars) is discussed. The cross section for scattering of charged mesons by a nucleon is computed, including the damping due to spin and charge inertia. Neglecting this damping, one obtains the results of the quantum mechanical treatment by elementary perturbation theory. *A. Pais.*

Kwal, Bernard. Une méthode d'approximations adaptée aux équations d'onde des corpuscules à mouvement intrinsèque (spin) et une nouvelle mécanique semi-classique de ces corpuscules. C. R. Acad. Sci. Paris **228**, 1634–1636 (1949).

Kwal, Bernard. Sur les équations de la théorie des champs spinoriels non localisables. C. R. Acad. Sci. Paris **230**, 276–278 (1950).

Kwal, Bernard. Les particules réciproques et la théorie des champs non localisables de Yukawa. C. R. Acad. Sci. Paris **230**, 184–186 (1950).

DECEMBER ISSUE IS AN INDEX WHICH HAS
BEEN PHOTOGRAPHED AT THE BEGINNING
OF THE VOLUME(S).